# 37784

OFFICE OF NAVAL RESEARCH

Contract N7onr-35801

T. O. I.

NR-041-032

# THE GENERALIZED SIMPLE WAVE

by

D. Naylor

DIVISION OF APPLIED MATHEMATICS

BROWN UNIVERSITY

PROVIDENCE, R. I.

June, 1954

All-111

# THE GENERALIZED SIMPLE WAVE[1]

by

D. Naylor[2]

Introduction. An important exact solution in the aerodynamics of steady, perfect and irrotational compressible flow is the Prandtl-Meyer expansion wave. This solution was originally obtained by seeking directly a solution of the equations of compressible flow referred to polar coordinates in which the radial distance is of no consequence to the density, pressure and magnitude and direction of the velocity. It therefore had the property that the density, pressure and velocity vector were constant along radii through a fixed point, but this restriction can be removed and the solution is capable of generalization to the case for which conditions are constant along a series of straight lines not necessarily concurrent. In the extended case the variables $u$, $v$, $p$, $\rho$ are all functions of just one independent variable and relations of the type $\partial(u,v)/\partial(x,y) = 0$ hold everywhere. The vectors $\nabla u$, $\nabla v$, $\nabla p$, $\nabla\rho$ are all parallel and each of the variables may be considered as reducing to a function of a function $\lambda(x,y)$ which may remain unspecified or chosen to reduce to $q$, or $\theta$ (say), or any combination of these quantities, as convenience requires. Such flows for which the velocity vector

1. The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract N7onr-35801 with Brown University.

2. Research Associate, Division of Applied Mathematics, Brown University.

$\underline{q}$ reduces to a (vector) function of just one independent variable $\lambda$ are now commonly referred to as "simple waves". It is known that steady irrotational flows defined in this way have the property that the pressure and density, as well as the velocity $\underline{q}$, reduce to functions of $\lambda$.

More generally it is desirable to extend the concept of the simple wave, for which $\underline{q}$ reduces to a function of $\lambda$ only, but to say nothing about the remaining variables p and $\rho$, to flows which are not necessarily steady and irrotational or for which conditions of velocity are not necessarily constant along straight lines or that the isovels intersect in a fixed point. The first step in this direction was taken by Prin [1]* who considered steady rotational plane flow by adapting the Prandtl-Meyer method thereby presupposing concurrent isovels. Martin, in a series of lengthy investigations, [2, 3, 4] succeeded in dealing with the more general case, without making any assumptions about the isovels, by extracting the simple wave from a general theory in which the pressure p and the stream density function $\psi$ were selected as fundamental variables. It appeared that the isovels constituted a system of straight lines, parallel or concurrent. Ghaffari [5] applied the original steady irrotational simple wave theory in terms of q as the fundamental variable with the aid of the Legendre contact transformation and Giese [6] later used this same procedure in terms of $\lambda$ to discuss essentially geometrical properties of the hodograph for one precisely

* Numbers in square brackets refer to the Bibliography at the end of the paper.

similar phenomena including, in addition to the simple wave $\underline{q}(\lambda)$, the "double" wave $\underline{q}(\lambda,\mu)$.

The One Parameter Fields. Only when the flow is unsteady is the velocity a function of two independent variables in which case the problem is mathematically a double wave, although the flow physically is instantaneously a simple wave varying with the time. The one parameter velocity field is a special application of a more general method in which any property of the flow may be regarded as being reducible to a function of only one parameter. In unsteady flow the method may be employed by focusing attention on velocity potentials of the type $\varphi(\lambda)$ or more generally $\varphi(\lambda,t)$. In linearized flow the method yields supersonic potentials when $\lambda^2 = x^2 - B^2(y^2 + z^2)$. The reduced potential of oscillatory flow is of this variety,

$$f(x,y,z) = \frac{\cos n_o\lambda}{\lambda}$$

with the standard notation $B^2 = M^2 - 1$, $n_o = \omega/aB^2$. The curves $\lambda(xyt)$ = constant, are the characteristic curves which in the generalized simple wave reduce in all cases to a system of straight lines (not necessarily coincident with one family of Mach lines of supersonic flow) except in the rotational unsteady compressible case.

The purpose of this report is to show how a variation of the hodograph method, based on the application of the Legendre contact transformation in terms of the parameter $\lambda$ which remains unspecified, may be conveniently employed to rapidly construct rotational and unsteady generalizations of the simple wave $\underline{q}(\lambda)$

when $\lambda = \lambda(x,y,t)$ in general. Finally the more general unsteady wave for which $\underline{q} = \underline{q}(\lambda;t)$ ; $\lambda = \lambda(xyt)$ is considered in Section IV.

<u>General Equations of the Flow.</u> The general equations of motion, continuity and adiabatic transformation,

$$\left(\frac{\partial}{\partial t} + \underline{q} \cdot \nabla\right)\underline{q} + \rho^{-1} \nabla p = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{q}) = 0$$

$$\left(\frac{\partial}{\partial t} + \underline{q} \cdot \nabla\right)(p\rho^{-\gamma}) = 0$$

may be transformed to give

$$\frac{\partial}{\partial t}\left(a^{\frac{2}{\gamma-1}}\right) + \nabla \cdot \left(\underline{q}\, a^{\frac{2}{\gamma-1}}\right) = 0 \qquad (1.1)$$

$$\frac{\partial \underline{q}}{\partial t} + \nabla H - T\nabla S = \underline{q}_\Lambda \underline{W} \qquad (1.2)$$

where

$$H = \frac{q^2}{2} + \frac{a^2}{\gamma - 1} \qquad S = C_v \log (p\rho^{-\gamma})$$

$$\underline{W} = \nabla_\Lambda \underline{q}$$

and $a = (\gamma p/\rho)^{1/2}$ defines the acoustic speed and $T = p/(\rho R)$ the absolute temperature. Since $T = a^2/\gamma R$ then Crocco's equation (1.2) may be used to yield another equation in addition to (1.1) involving just $a^2$ and $\underline{q}$

$$\nabla_\Lambda \left\{a^{-2} \left(\frac{\partial}{\partial t} + \underline{q} \cdot \nabla\right)\underline{q}\right\} = 0 \qquad (1.3)$$

or

$$\nabla_\Lambda \left\{a^{-2} \left(\underline{q}_t + q\nabla q - \underline{q}_\Lambda \underline{W}\right)\right\} = 0. \qquad (1.4)$$

Steady Rotational Flows; A Simplified Theory. Attention being confined to iso-energetic flows for which H = constant, the general equations for the velocity reduce to

$$\nabla \cdot (\underline{q}\, a^{\frac{2}{\gamma-1}}) = 0 \qquad (1.5)$$

$$\nabla_\Lambda (a^{-2}\, \underline{q}_\Lambda \underline{W}) = 0 \qquad (1.6)$$

where

$$\frac{q^2}{2} + \frac{a^2}{\gamma - 1} = \frac{c^2}{2} \qquad (1.7)$$

serves to provide the acoustic speed directly in terms of the velocity and the iso-energetic constant c (actually the greatest possible speed). The fundamental assumption of the existence of a flow of the simple wave pattern is most easily exploited by using, not the ordinary stream function, but Crocco's stream function $\psi$ given by

$$\nabla \psi = \underline{K}_\Lambda \underline{q}\, a^{\frac{2}{\gamma-1}} \qquad (1.8)$$

the necessary condition for the existence of which is implied by (1.5), and then define the associated function $\Psi(x,y)$ by the Legendre contact transformation

$$\psi = \Psi + \underline{r} \cdot \underline{Q} \qquad (1.9)$$

with the notation $\underline{Q} = \underline{K}_\Lambda \underline{q}\, a^{\frac{2}{\gamma-1}} = \nabla\psi$ which in the following analysis is convenient because $\underline{Q}$ rather than $\underline{q}$, is the fundamental field vector.

The transformations employed are two-fold:

1. The current independent variables (x,y) are replaced by $(\lambda,\psi)$,

2. The dependent function $\psi(x,y)$ is itself replaced by $\Psi$, its Legendre transform, but is also retained as a fundamental independent variable where necessary.

This procedure reduces the general problem of the rotational simple wave, originally that of two non-linear partial differential equations for $\underline{q}(\lambda)$; $w(x,y)$ to the study of an ordinary second order differential equation for $\underline{q}(\lambda)$ and a linear ordinary second order differential equation for $\Psi(\lambda)$. This reduction is rapidly achieved because $\underline{Q}$ and $\Psi$ both reduce to functions of $\lambda$ only, they are not general functions of the new variables $(\lambda,\psi)$, as for example, is $w(\lambda,\psi)$. The transformation $(x,y) \longrightarrow (\lambda,\psi)$ is permissible because

$$\frac{\partial(\lambda,\psi)}{\partial(x,y)} \neq 0$$

which independence is immediate because

$$\nabla_{\Lambda} \underline{Q}(\lambda) = - \underline{Q}_{\lambda \Lambda} \nabla\lambda = 0$$

in virtue of $\nabla_{\Lambda} \nabla\psi \equiv 0$.

$$\begin{array}{ll} \text{It follows that} & \nabla\lambda_{\Lambda} \nabla\psi = 0, \\ \text{that is,} & \underline{Q}_{\Lambda} \nabla\lambda = 0 \\ \text{only when} & \underline{Q}_{\Lambda} \underline{Q}_{\lambda} = 0 \end{array}$$

which is the excluded degenerate case, $\Theta_{\lambda} = 0$, of a parallel (shear) flow. The following outlines and illustrates the general method, the principles of which are subsequently adapted to study unsteady flows in the sections which follow. Full details and results of the rotational one-parameter fields will be found

in the author's memoir [7] where a slightly different treatment is employed. It is emphasized that the adopted procedure is not restricted to flows of the type here discussed, but with modification is capable of application to a great diversity of fluid flow phenomena of the simple wave pattern, (e.g. unsteady, rotational and viscous incompressible motion - see another report), the criterion being the existence of a suitable field gradient vector depending on $\lambda$ only. It is at once clear from (1.7), and (1.8) that $a^2$, and $\underline{Q}$, reduce to functions of $\lambda$ only, being independent of $\psi$, therefore it is possible to write the total differential of (1.9) in the form

$$d\Psi + (\underline{r} \cdot \underline{Q}_\lambda)d\lambda = 0$$

so that $\Psi = \Psi(\lambda)$ and

$$\Psi_\lambda + \underline{r} \cdot \underline{Q}_\lambda = 0 \tag{1.10}$$

from which by further differentiation is obtained

$$(\Psi_{\lambda\lambda} + \underline{r} \cdot \underline{Q}_{\lambda\lambda})\nabla\lambda + \underline{Q}_\lambda = 0 \tag{1.11}$$

The following representations are employed,

$$|\underline{q}_{\lambda\Lambda}\ \underline{Q}_\lambda| = D(\lambda)\ a^{\frac{2\gamma}{\gamma-1}} \tag{1.12}$$

$$\underline{Q}_{\lambda\lambda} + F(\lambda)\underline{Q}_\lambda = K(\lambda)\underline{Q} \tag{1.13}$$

where

$$a^4 D(\lambda) = (a^2 - q^2)q_\lambda^2 + a^2q^2\theta_\lambda^2 \tag{1.14}$$

$$F(\lambda) = -\frac{d}{d\lambda}(\log \theta_\lambda) \tag{1.15}$$

$$K(\lambda) = \frac{Q^3\theta_\lambda^2}{2Q_\lambda}\frac{d}{d\lambda}[Q^{-2} + \frac{Q_\lambda^2}{Q^4\theta_\lambda^2}] \tag{1.16}$$

Also an expression for the vorticity which follows from (1.6), and (1.8)

$$\nabla\psi_{\wedge} \nabla(w\, a^{-\frac{2\gamma}{\gamma-1}}) = 0$$

or

$$w = G(\psi) a^{\frac{2\gamma}{\gamma-1}} \tag{1.17}$$

in terms of some function $G(\psi)$.

Since $\underline{w} = \nabla_{\wedge} \underline{q} = \nabla\lambda_{\wedge} \underline{q}_{\lambda}$ then (1.11), (1.12), and (1.17) gives

$$(\Psi_{\lambda\lambda} + \underline{r} \cdot \underline{Q}_{\lambda\lambda}) \; G\,(\psi) = D(\lambda)$$

finally substitution of the scalar produce $\underline{r} \cdot \underline{Q}_{\lambda\lambda}$ from (1.13) with the aid of (1.9), and (1.10) gives

$$[\Psi_{\lambda\lambda} + F\Psi_{\lambda} + K(\psi - \Psi)]\; G\,(\psi) = D(\lambda) \tag{1.18}$$

Since $(\lambda, \psi)$ are independent, a field equation of the type (1.18) can only be true (since F, K, $\Psi$, D are independent of $\psi$) if either

1. $G(\psi) = A$; $\quad K(\lambda) = 0$; $\quad \Psi_{\lambda\lambda} + F(\lambda)\Psi_{\lambda} = \frac{D(\lambda)}{A}$
2. $G(\psi) = A/\psi$; $AK(\lambda) = D(\lambda)$; $\quad \Psi_{\lambda\lambda} + F(\lambda)\Psi_{\lambda} - K(\lambda)\Psi = 0$

where A is a constant determining the vorticity distribution in alternative forms

$$A\, a^{\frac{2\gamma}{\gamma-1}}; \qquad (A/\psi)\, a^{\frac{2\gamma}{\gamma-1}}$$

in terms of Crocco's stream function $\psi$; if formulae expressed in terms of the normal stream function are required then appropriate conversion relations must be employed.

<u>Properties of the $\lambda$-lines.</u> The first possibility $K(\lambda) = 0$ implies, from (1.13),

$$\underline{Q}_{\lambda\lambda} + F(\lambda)\underline{Q}_{\lambda} = 0$$

which may be integrated to show that $\underline{Q}_{\lambda}$ is parallel to a fixed vector. Therefore the $\lambda$-lines (1.10) constitute a system of parallel lines. These are the flows which appear to elude treatment by other general methods and for which a direct approach is simpler.

The second possibility, for which

$$\Psi_{\lambda\lambda} + F(\lambda)\Psi_{\lambda} - K(\lambda)\Psi = 0 \tag{1.19}$$

$$\underline{Q}_{\lambda\lambda} + F(\lambda)\underline{Q}_{\lambda} - K(\lambda)\underline{Q} = 0 \tag{1.20}$$

implies, eliminating the functions $F(\lambda)$, $K(\lambda)$,

$$\begin{vmatrix} \Psi & U & V \\ \Psi_{\lambda} & U_{\lambda} & V_{\lambda} \\ \Psi_{\lambda\lambda} & U_{\lambda\lambda} & V_{\lambda\lambda} \end{vmatrix} = 0$$

where $(U,V)$ are the components of $\underline{Q}(\lambda)$. This means however that the three non-parallel lines

$$\Psi + \underline{r} \cdot \underline{Q} = 0; \quad \Psi_{\lambda} + \underline{r} \cdot \underline{Q}_{\lambda} = 0, \quad \Psi_{\lambda\lambda} + \underline{r} \cdot \underline{Q}_{\lambda\lambda} = 0$$

are concurrent and implies the existence of a vector $\underline{r}_o(\lambda)$ such that

$$\Psi + \underline{r}_o \cdot \underline{Q} = 0; \quad \Psi_{\lambda} + \underline{r}_o \cdot \underline{Q}_{\lambda} = 0; \quad \Psi_{\lambda\lambda} + \underline{r}_o \cdot \underline{Q}_{\lambda\lambda} = 0$$

identically in $\lambda$.

Differentiating the first one and using the second gives

$$(\underline{r}_o)_\lambda \cdot \underline{Q} = 0$$

whilst differentiating the second and using the third yields

$$(\underline{r}_o)_\lambda \cdot \underline{Q}_\lambda = 0$$

so that since the vectors $\underline{Q}$, $\underline{Q}_\lambda$ are neither zero nor parallel (except in degenerate cases),

$$\frac{d\underline{r}_o}{d\lambda} = 0.$$

Therefore $\underline{r}_o$ is a fixed vector, independent of $\lambda$, and the three lines in question all pass through the same fixed point for all values of $\lambda$; in particular the parameter lines (1.10) are concurrent. These results show that in no case is a non-degenerate envelope of the $\lambda$-lines possible, and also that in the case of concurrent $\lambda$-lines, that, by selecting as the origin the point through which they all pass, $\Psi(\lambda) = 0$. In other words, the Legendre transform of Crocco's stream function may be taken to be identically zero in the theory of a rotational steady simple wave, except when conditions are constant along a series of straight lines. This is in direct contrast to the classical simple wave theory in steady potential flow for which an arbitrary convex $\lambda$-envelope is possible corresponding to arbitrary $\Psi(\lambda)$. The preceding method is still valid, however, as far as equation (1.18), which in the simpler case of $G(\psi) = 0$ simply reduces to $D(\lambda) = 0$ so that $\Psi(\lambda)$ may be arbitrarily chosen (subject to the condition that the $\lambda$-envelope is a simple convex curve). By

(1.14) the equation for the velocity $D(\lambda) = 0$ will yield by integration the standard hodograph epicycloid $\Theta = \Theta(q)$. More directly (1.5) gives

$$\nabla \lambda \cdot (\underline{q}\, a^{\frac{2}{\gamma-1}})_\lambda = 0$$

and $\nabla_\Lambda \underline{q} = 0$ gives

$$\nabla \lambda_\Lambda \underline{q}_\lambda = 0$$

therefore the velocity relation for a classical simple wave is

$$\underline{q}_\lambda \cdot (\underline{q}\, a^{\frac{2}{\gamma-1}})_\lambda = 0$$

which on expansion yields $D(\lambda) = 0$.

2. The Simple Wave in Unsteady Flow. When unsteady flow is considered a stream function never exists in compressible flow and it is necessary to confine attention to irrotational motion in order to adapt the preceding method by replacing the stream function of Crocco $\psi$ by the velocity potential function $\varphi(x,y,t)$. The equations of motion, continuity, irrotational motion and isentropic flow are

$$\rho_t + \nabla \cdot (\rho \underline{q}) = 0$$

$$\underline{q}_t + (\underline{q} \cdot \nabla)\underline{q} + \frac{1}{\rho} \nabla p = 0$$

$$\nabla_\Lambda \underline{q} = 0$$

$$p\rho^{-\gamma} = \text{constant}$$

may be used to yield the general equations of unsteady potential flow in the form

$$\frac{\partial}{\partial t}\,(a^b) + \nabla \cdot (\underline{q}\; a^b) = 0 \qquad (2.1)$$

$$\frac{q^2}{2} + \frac{a^2}{\gamma-1} + \frac{\partial\varphi}{\partial t} = \frac{c^2}{2} \qquad (2.2)$$

where c is a constant, $b = 2/\gamma\text{-}1$, $\underline{q} = \bar{\nabla}\varphi$, and $a = (\frac{\gamma p}{\rho})^{1/2}$ is the acoustic speed.

Introducing $\Phi$, the Legendre transform of $\varphi$,

$$\varphi = \Phi + \underline{r} \cdot \nabla\varphi \qquad (2.3)$$

then

$$d\Phi + \underline{r} \cdot d\underline{q} = 0$$

The simplest generalization of the simple wave will be one for which the velocity vector $\underline{q}$ reduces to a function of just one independent parameter $\lambda$ incorporating the time, $\lambda = \lambda(\underline{r}\ ;\ t)$. Therefore $\underline{q} = \underline{q}(\lambda)$ and $d\underline{q} = \underline{q}_\lambda d\lambda$ so that

$$d\Phi + \underline{r} \cdot \underline{q}_\lambda d\lambda = 0$$

$$\nabla\Phi + (\underline{r} \cdot \underline{q}_\lambda)\nabla\lambda = 0$$

therefore $\Phi = \Phi(\lambda, t)$ and

$$\Phi_\lambda + \underline{r} \cdot \underline{q}_\lambda = 0 \qquad (2.4)$$

the function $\Phi$ reducing to a function of $(\lambda, t)$ only, involving t independently as well as through $\lambda(\underline{r}\ ;\ t)$.

The surfaces of constant $\lambda$, that is of constant $\underline{q}$, constitute a system of planes which vary with the time. In plane flow the velocity is constant along each of a family of straight lines varying with the time and normal to $\underline{q}_\lambda$. The angle of intersection $\Gamma(\lambda)$ with the streamlines being given by

$$\cot\Gamma = (q\theta_\lambda)/q_\lambda \qquad (2.5)$$

Differentiation of (2.4) with respect to a position gives

$$(\Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda})d\lambda + d\underline{r} \cdot \underline{q}_{\lambda} = 0$$

that is, since $d\lambda = d\underline{r} \cdot \nabla\lambda$,

$$(\Phi_{\lambda\lambda} + \underline{r} \cdot q_{\lambda\lambda})\nabla\lambda + \underline{q}_{\lambda} = 0 \qquad (2.6)$$

Differentiating (2.3) partially with respect to the time gives, since $\Phi = \Phi(\lambda;t)$

$$\dot{\varphi} = \Phi_t + (\Phi_\lambda + \underline{r} \cdot \underline{q}_\lambda)\dot{\lambda}$$

that is

$$\dot{\varphi} = \Phi_t \qquad (2.7)$$

in virtue of the field equation (2.4). An additional differentiation now gives

$$\ddot{\varphi} = \Phi_{tt} + \Phi_{\lambda t}\dot{\lambda}$$

whilst the expression for $\dot{\lambda}$ analogous to (2.6) for $\nabla\lambda$ is similarly obtained from (2.4) by differentiation with respect to the time giving

$$(\Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda})\dot{\lambda} + \Phi_{\lambda t} = 0 \qquad (2.8)$$

so that the final expression for $\ddot{\varphi}$ is

$$\ddot{\varphi} = \Phi_{tt} - \Phi_{\lambda t}^2(\Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda})^{-1} \qquad (2.9)$$

For flows of the type considered we are replacing the fundamental independent variables $(x,y,t)$ by $(\lambda,\varphi,t)$ so that the vector operator for grad,

$$\nabla = \nabla\lambda \frac{\partial}{\partial\lambda} + \nabla\varphi \frac{\partial}{\partial\varphi}$$

are equivalent. When operating on $\underline{q}(\lambda)$ this operator reduces to $\nabla\lambda\ \partial/\partial\lambda$; but the general form must be retained for functions like $\dot{\varphi}$, $\ddot{\varphi}$ and $a^2 = \gamma p/\rho$, since they will in general be functions of $\varphi$ as well as $(\lambda,t)$.

Actually in the one parameter fields here considered only $\ddot{\varphi}$ does not in general reduce to a function of $(\lambda,t)$ only.

The equations (2.1) and (2.2) for the velocity now become

$$(a^2\underline{q}_\lambda - qq_\lambda\underline{q}) \cdot \nabla\lambda = 2qq_\lambda\dot{\lambda} + \ddot{\varphi}$$

$$\frac{q^2}{2} + \frac{a^2}{\gamma-1} + \Phi_t = \frac{c^2}{2} \qquad (2.10)$$

and the first gives, on substituting for $\nabla\lambda$, $\dot{\lambda}$ and $\ddot{\varphi}$,

$$(a^2\underline{q}_\lambda - qq_\lambda\underline{q}) \cdot \underline{q}_\lambda = (2qq_\lambda + \Phi_{\lambda t})\Phi_{\lambda t} - \Phi_{tt}(\Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda})$$

that is

$$a^2\underline{q}_\lambda^2 = (qq_\lambda + \Phi_{\lambda t})^2 - \Phi_{tt}(\Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda}) \qquad (2.11)$$

Since the ultimate aim is to replace $(x,y)$ by $(\lambda,\varphi)$, it remains to obtain suitable expressions for the scalar product $\underline{r} \cdot \underline{q}_{\lambda\lambda}$ occurring in (2.11). This is achieved by noting that, except in degenerate cases, the original field equations defining $\lambda(x,y,t)$, $\Phi(x,y,t)$,

$$\Phi + \underline{r} \cdot \underline{q} = \varphi, \qquad \Phi_\lambda + \underline{r} \cdot \underline{q}_\lambda = 0 \qquad (2.12)$$

may be solved explicitly for $\underline{r}$ to yield a parametric representation of the plane in the form $\underline{r}(\lambda,\varphi,t)$. The position vector of the field point occurring in (2.11) may be replaced as required by functions of the independent variables $(\lambda,\varphi,t)$ by using the expression

$$\underline{q}_{\lambda\lambda} + A_1(\lambda)\underline{q} + B_1(\lambda)\underline{q}_\lambda = 0 \tag{2.13}$$

where

$$\left.\begin{aligned} A_1(\lambda) &= -\frac{q^3\theta_\lambda^2}{2q_\lambda}\frac{d}{d\lambda}[q_\lambda^2\, q^{-4}\theta_\lambda^{-2} + q^{-2}] \\ B_1(\lambda) &= -\frac{d}{d\lambda}\log q^2\theta_\lambda \end{aligned}\right\} \tag{2.14}$$

Scalar multiplication of (2.13) by $\underline{r}$ and using (2.12) gives finally

$$\underline{r}\cdot\underline{q}_{\lambda\lambda} = (\Phi - \varphi)A_1(\lambda) + \Phi_\lambda B_1(\lambda)$$

so that (2.11) becomes

$$a^2\underline{q}_\lambda^2 = (qq_\lambda + \Phi_{\lambda t})^2 - \Phi_{tt}[\Phi_{\lambda\lambda} + \Phi_\lambda B_1 + A_1(\Phi - \varphi)] \tag{2.15}$$

in which $a^2$, being given by (2.10), is independent of $\varphi$.

$\varphi(r;t)$ never reduces to a function of $(\lambda,t)$ only for otherwise the velocity vector

$$\underline{q} = \nabla\varphi = \varphi_\lambda\nabla\lambda$$

will only give irrotational flow

$$\underline{w} = \nabla_\Lambda \underline{q} = \nabla\lambda_\Lambda \underline{q}_\lambda = 0$$

in the trivial case $\underline{q}_\Lambda \underline{q}_\lambda = 0$. It follows that if the field equation (2.15) is to be valid everywhere then

$$A_1(\lambda)\Phi_{tt}(\lambda,t) = 0 \tag{2.16}$$

and

$$a^2\underline{q}_\lambda^2 = (qq_\lambda + \Phi_{\lambda t})^2 - \Phi_{tt}(\Phi_{\lambda\lambda} + B_1\Phi_\lambda) \tag{2.17}$$

The condition (2.16) shows at once that either, or both, of $A_1(\lambda)$, $\Phi_{tt}(\lambda,t)$ must vanish identically.

I. $\Phi_{tt}(\lambda,t) = 0$. $\Phi_t(\lambda,t)$ reduces to a function of $\lambda$ only say $g(\lambda)$ and we can write

$$\Phi(\lambda,t) = tg(\lambda) + f(\lambda) \tag{2.17}$$

where $g(\lambda)$ is related to $\underline{q}(\lambda)$ by (2.17),

$$a^2 \underline{q}_\lambda^2 = [qq_\lambda + g'(\lambda)]^2 \tag{2.18}$$

and

$$\frac{q^2}{2} + \frac{a^2}{\gamma-1} + g(\lambda) = \frac{c^2}{2} \tag{2.19}$$

It follows the parameter $\lambda(x,y,t)$ is given, implicitly, by the field equation (2.4),

$$tg'(\lambda) + f'(\lambda) + \underline{r} \cdot \underline{q}_\lambda = 0 \tag{2.20}$$

Regarding f and g as functions of q the following relation is equivalent to (2.18), and (2.19),

$$(\gamma - 1)(1 + q^2\theta_q^{\,2})(c^2 - q^2 - 2g) = 2(q + g_q)^2 \tag{2.21}$$

The functions $f(q)$; $g(q)$ may therefore be chosen arbitrarily, the velocity distribution $\theta(q)$ corresponding to any selected $g(q)$ being obtained from the ordinary differential equation (2.21). The field equation (2.20) may then be used to determine $\theta(q)$ as a function of $(x,y,t)$

II. $A_1(\lambda) = 0$. The second possibility implies from (2.13), and (2.14)

$$\underline{q}_{\lambda\lambda} - \underline{q}_\lambda\left(\frac{d}{d\lambda}\log q^2\theta_\lambda\right) = 0$$

that is

$$\frac{d}{d\lambda}\left(\frac{1}{q^2\theta_\lambda}\,\underline{q}_\lambda\right) = 0$$

so that $(1/q^2\theta_\lambda)\underline{q}_\lambda$ is a constant vector. Therefore the $\lambda$-lines (2.4) comprise a system of lines parallel to a fixed direction. Integration of $A_1(\lambda) = 0$ twice, using (2.14), gives

$$q = q_o \operatorname{cosec} (\theta + \theta_o) \tag{2.22}$$

in terms of constants of integration $q_o$, $\theta_o$. The component of velocity in the direction $\theta = \frac{\pi}{2} - \theta_o$ has the constant value $q_o$. Since

$$\underline{q} \sin (\theta + \theta) = q_o \, \underline{t}(\theta)$$

with the usual notation for $\underline{t}(\theta)$ the unit tangent vector to the streamline. Then

$$\underline{q}_\theta \sin^2(\theta + \theta_o) = - q_o \, \underline{t}(\theta_o)$$

therefore the $\lambda$-lines

$$\Phi_\theta + \underline{r} \cdot \underline{q}_\theta = 0$$

may be rewritten

$$\Phi_\theta \sin^2(\theta + \theta_o) = q_o \, \underline{r} \cdot \underline{t}(- \theta_o) \tag{2.23}$$

When expressed in terms of $\theta$ the equation (2.17) becomes

$$a^2\underline{q}_\theta^2 = (qq_\theta + \Phi_{\theta t})^2 - \Phi_{tt}\, q^2 \frac{d}{d\theta} (q^{-2}\Phi_\theta)$$

and then, using the parameter $z = q_o \cot (\theta + \theta_o)$, may be transformed into the form

$$a^2 = (z + \Phi_{zt})^2 - \Phi_{zz}\, \Phi_{tt}$$

where

$$\frac{2a^2}{\gamma-1} = c^2 - q_o^2 - z^2 - 2\Phi_t$$

or finally using

$$F(z;t) = \Phi(z;t) + \tfrac{1}{2}\,tz^2 + \tfrac{1}{2}\,(q_o^2 - c^2)t \tag{2.24}$$

so that

$$a^2 = -\,(\gamma - 1)F_t(z;t) \tag{2.25}$$

then

$$F_{zt}^{\;2} - F_{zz}F_{tt} + tF_{tt} + (\gamma - 1)F_t = 0 \tag{2.26}$$

Similarly the equation (2.23) for the parametric lines may be expressed in terms of z and F(z,t),

$$F_z(z;t) - tz + x\cos\theta_o - y\sin\theta_o = 0$$

which serves to provide z as a function of (x,y,t) when any solution of (2.26) for F(z;t) is inserted. If $\theta_o = 0$ is selected then the parameter lines are parallel to the y-axis and are given by

$$x = tz - F_z(z;t)$$

where $z = q_o \cot\theta$.

<u>Concurrent Parameter Lines</u>. When the classical theory of the steady irrotational simple wave is developed, the general simple wave generated by λ-lines enveloping a convex curve is obtained by a generalization of the original Prandtl-Meyer expansion wave for which the λ-lines are concurrent through a fixed point. It is clear however that this procedure cannot be repeated to obtain a generalized unsteady simple wave from the case of concurrent parameter lines, for this latter is the very case which is automatically excluded by (2.17). For if the λ-lines are concurrent through a <u>fixed</u> point, this may be selected to coincide with the origin then (2.20) gives $g'(\lambda) = f'(\lambda) = 0$ so that $\Phi$ reduces to a function of time only, in fact $\Phi = at + b$

where a,b are constants. The equation for the velocity (2.18) reduces to

$$a^2 \underline{q}_\lambda^2 = q^2 q_\lambda^2$$

where

$$\frac{q^2}{2} + \frac{a^2}{\gamma - 1} = \text{constant}$$

so that the entire flow has reduced to the Prandtl-Meyer simple centered expansion wave in steady flow. The second possibility $A_1(\lambda) = 0$ implying parallel $\lambda$-lines (preceding pages), it follows that there is no unsteady simple centered wave, with fixed center. of the type $\underline{q}(\lambda)$.

The General Case. The envelope E(t) will be obtained by solving for $\underline{r}$ the equations

$$\left.\begin{aligned} \Phi_\lambda + \underline{r} \cdot \underline{q}_\lambda &= 0 \\ \Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda} &= 0 \end{aligned}\right\} \qquad (2.27)$$

the second, representing a one parameter family of straight lines, being obtained from the first by partial differentiation with respect to $\lambda$ (x,y,t assumed constant). These lines will be parallel if $\underline{q}_{\lambda\Lambda}\, \underline{q}_{\lambda\lambda} = 0$, or $A_1(\lambda) = 0$, which exceptional case shall be excluded at present. For any given $(\lambda, t)$, the system (2.27) represents a line pair intersecting in the point $\underline{r}_o(\lambda;t)$ given by

$$\underline{r}_o(\lambda;t) = (\Phi_\lambda \underline{Q}_{\lambda\lambda} - \Phi_{\lambda\lambda}\, \underline{Q}_\lambda) \,|\, \underline{q}_{\lambda\Lambda}\, \underline{q}_{\lambda\lambda} |^{-1}$$

Now vector multiplication of (2.13) by $\underline{q}_\lambda$ gives, since

$$|\underline{q}_{\Lambda}\ \underline{q}_{\lambda}| = q^2\theta_{\lambda} = \exp - \int B_1(\lambda)d\lambda,$$

then

$$|\underline{q}_{\lambda\Lambda}\ \underline{q}_{\lambda\lambda}| = A_1(\lambda) \exp - \int B_1(\lambda)d\lambda$$

therefore

$$A_1(\lambda)\underline{r}_o(\lambda;t) = (\Phi_{\lambda}\ \underline{Q}_{\lambda\lambda} - \Phi_{\lambda\lambda}\ \underline{Q}_{\lambda}) \exp \int B_1(\lambda)d\lambda \qquad (2.28)$$

As $\lambda$ varies, $\underline{r}_o(\lambda,t)$ will, for fixed t, generate a curve, the envelope E(t) of the $\lambda$-lines. If the $\lambda$-lines are concurrent then E(t) degenerates into a point and $\underline{r}_o(\lambda;t)$ is fixed for all values of $\lambda$. That is $\underline{r}_o$ reduces to a function of time only, therefore $\partial\underline{r}_o/\partial\lambda = 0$.

Elimination of the vector $\underline{Q}_{\lambda\lambda}$ from (2.13) gives

$$A_1(\lambda)\underline{r}_o(\lambda;t) = - [\Phi_{\lambda\lambda}\underline{Q}_{\lambda} + \Phi_{\lambda}\ (A_1\underline{Q} + B_1\underline{Q}_{\lambda})] \exp \int B_1(\lambda)d\lambda$$

giving after partial differentiation with respect to $\lambda$ and repeated use of (2.13),

$$A_1(\lambda)\ \underline{r}_{o\lambda} = - [\Phi_{\lambda\lambda\lambda} + B_1\Phi_{\lambda\lambda} + (A_1 + B_{1\lambda})\Phi_{\lambda}$$
$$- \frac{A_{1\lambda}}{A_1}(\Phi_{\lambda\lambda} + B_1\ \Phi_{\lambda})]\underline{Q}_{\lambda} \times \exp \int B_1(\lambda)d\lambda$$

so that $(\underline{r}_o)_{\lambda} = 0$ when $\Phi(\lambda;t)$ satisfies

$$\Phi_{\lambda} + \frac{1}{A_1} \cdot \frac{\partial}{\partial\lambda}(\Phi_{\lambda\lambda} + B_1\Phi_{\lambda}) - (\Phi_{\lambda\lambda} + B_1\Phi_{\lambda})\frac{A_{1\lambda}}{A_1^2} = 0$$

giving, on partial integration in terms of a function of integration h(t),

$$\Phi_{\lambda\lambda} + B_1(\lambda)\Phi_{\lambda} + A_1(\lambda)\Phi = h(t)\ A_1(\lambda)$$

Since the case $A_1(\lambda) = 0$ is excluded, then $\Phi(\lambda;t)$ is restricted to the special form (2.17) so that

$$(g_{\lambda\lambda} + B_1 g_\lambda + A_1 g) + (f_{\lambda\lambda} + B_1 f_\lambda + A_1 f)t = A_1(\lambda)h(t)$$

in terms of $g(\lambda)$, $f(\lambda)$, $A_1(\lambda)$, $B_1(\lambda)$ which are not functions of the independent variable t. It follows that h(t) is not arbitrary but must be a linear function of t, defined in terms of constants $(a_o, b_o)$, not necessarily non-zero,

$$h(t) = a_o + tb_o$$

so that $f(\lambda)$, $g(\lambda)$ are now given by

$$g_{\lambda\lambda} + B_1 g_\lambda + A_1(g - a_o) = 0$$

$$f_{\lambda\lambda} + B_1 f_\lambda + A_1(f - b_o) = 0$$

Therefore, apart from additive constants not affecting $\Phi_\lambda(\lambda;t)$, the functions $f(\lambda)$, $g(\lambda)$ both satisfy the ordinary linear differential equation

$$g_{\lambda\lambda} + B_1 g_\lambda + A_1 g = 0 \tag{2.29}$$

in order that the $\lambda$-lines may be instantaneously concurrent. When this condition is satisfied the time variation of $\underline{r}_o(t)$ will be given by (2.28); since $\Phi_t = g(\lambda)$,

$$A_1(\lambda)\ (\underline{r}_o)_t = (g_\lambda\, \underline{Q}_{\lambda\lambda} - g_{\lambda\lambda}\, \underline{Q}_\lambda) \exp \int B_1(\lambda)d\lambda$$

$$= -\ [g_\lambda(A_1\, \underline{Q} + B_1\, \underline{Q}_\lambda) + g_{\lambda\lambda}\, \underline{Q}_\lambda] \exp \int B_1(\lambda)d\lambda$$

or

$$(\underline{r}_o)_t = [g\, \underline{Q}_\lambda - g_\lambda\, \underline{Q}] \exp \int B_1(\lambda)d\lambda$$

in virtue of (2.29). $\underline{r}_o'(t)$ being independent of t or $\lambda$; it follows

that the point of concurrency (the instantaneous "center" of the wave) moves with constant velocity in a fixed direction. [Although apparently a simple function of $\lambda$, it may be demonstrated by direct differentiation with respect to $\lambda$ and the use of (2.29); (2.13) that $(\underline{r}_o)_{\lambda t} = 0$, as it must since $(\underline{r}_o)_\lambda = 0$, so that $(\underline{r}_o)_t$ reduces to a constant.]

Therefore if

$$\Phi_{\lambda\lambda} + B_1(\lambda)\Phi_\lambda + A_1(\lambda)\Phi = 0 \tag{2.30}$$

the $\lambda$-lines are concurrent through the point $\underline{r}_o(t)$

$$A_1(\lambda)\,\underline{r}_o(t) = (\Phi_\lambda\,\underline{Q}_{\lambda\lambda} - \Phi_{\lambda\lambda}\underline{Q}_\lambda)\exp\int B_1(\lambda)d\lambda$$

which with the aid of (2.13), and (2.30) reduces to

$$\underline{r}_o(t) = (\Phi\,\underline{Q}_\lambda - \Phi_\lambda\,\underline{Q})\exp\int B_1(\lambda)d\lambda \tag{2.31}$$

which, in fact, is also the intersection of the lines

$$\Phi + \underline{r}\cdot\underline{q} = 0; \qquad \Phi_\lambda + \underline{r}\cdot\underline{q}_\lambda = 0$$

The three lines

$$\Phi + \underline{r}\cdot\underline{q} = 0; \quad \Phi_\lambda + \underline{r}\cdot\underline{q}_\lambda = 0; \quad \Phi_{\lambda\lambda} + \underline{r}\cdot\underline{q}_{\lambda\lambda} = 0$$

are therefore concurrent, for all $\lambda$, through the point $\underline{r}_o(t)$. This property may also be derived by a similar method to that resorted to for the steady rotational flows; equations (2.13), and (2.30) being exactly analogous to (1.20) and (1.19).

When (2.30) is satisfied the unsteady simple wave reduces to a flow which is essentially "steady". In fact a simple

transformation to the moving rectangular coordinate system $\underline{r}'$

$$\underline{r}'(\lambda,\varphi,t) = \underline{r}(\lambda,\varphi,t) - \underline{r}_o(t)$$

gives since $\underline{r}(\lambda,\varphi,t) = [(\Phi - \varphi)\underline{Q}_\lambda - \Phi_\lambda \underline{Q}] \exp \int B_1(\lambda)d\lambda$

obtained by solving (2.3), and (2.4), therefore

$$\underline{r}'(\lambda,\varphi,t) = - \varphi \underline{Q}_\lambda \exp \int B_1(\lambda)d\lambda$$

so that the revised representation reduces to $\underline{r}'(\lambda,\varphi)$ and the equation of the parameter line $\Phi_\lambda + \underline{r} \cdot \underline{q}_\lambda = 0$ referred to the moving axes is

$$\Phi_\lambda + (\underline{r}' + \underline{r}_o) \cdot \underline{q}_\lambda = 0$$

in which (2.31) gives

$$\Phi_\lambda + \underline{r}_o \cdot \underline{q}_\lambda = \Phi_\lambda - \Phi_\lambda (\underline{Q} \cdot \underline{q}_\lambda) \exp \int B_1(\lambda)d\lambda$$
$$= 0$$

in virtue of the definition (2.14) of $B_1(\lambda)$. The fundamental equation of the lines of constant $\lambda$ therefore reduces to

$$\underline{r}' \cdot \underline{q}_\lambda = 0$$

which, since $\underline{q} = \underline{q}(\lambda)$, defines $\lambda$ as a function of $\underline{r}'$ only, not t. Referred to the new coordinate system moving along with the point of concurrency $\underline{r}_o(t)$ with constant velocity, therefore, the flow reduces to a steady phenomenon - the simple central Prandtl-Meyer expansion wave.

Non-Degenerate Envelope. The case of concurrent λ-lines representing no essentially new phenomena, attention will be directed to flows possessing a non-degenerate λ-envelope. For such flows the functions f, g may be regarded as functions of q (this amounts to selecting the arbitrary parameter λ(xyt) to coincide with q(xyt) = q(λ) = q = λ). The relevant equations are

$$(\gamma - 1)(1 + q^2\theta_q^{\,2})(c^2 - q^2 - 2g) = 2(q + g_q)^2 \qquad (2.32)$$

$$tg_q + f_q + \underline{r} \cdot \underline{q}_q = 0 \qquad (2.33)$$

then, provided at least one of the relations

$$g_{qq} + B_1(q)g_q + A_1(q)g \neq 0$$

$$f_{qq} + B_1(q)f_q + A_1(q)f \neq 0$$

is not violated, the λ-lines will not be concurrent.

Solution 1. Take for example $g(q) = \frac{1}{2} q^2$; $f(q) = 0$ then,

if $b^2 = c^2/2$

$$a^2 = (\gamma - 1)(b^2 - q^2)$$

let

$$q^2 = b^2(\sin^2 z + \rho^2\cos^2 z)$$

$$\rho^2 = \frac{\gamma - 1}{\gamma + 3}$$

then

$$a = 2b\rho \cos z$$

so that, from (2.32),

$$q\theta_q = \rho^{-1} \tan z \qquad (2.34)$$

when the positive sign for the square root is selected.

Expressed in terms of z,

$$\rho\Theta_z = (1 - \rho^2)(1+\rho^2\cot^2 z)^{-1}$$

which may be integrated to give

$$\Theta = \rho^{-1}z + \tan^{-1}(\rho \cot z) + \pi - \frac{\pi}{2\rho} \qquad (2.35)$$

by suitable choice of the constant of integration.

Now from (2.5) the angle which the streamlines make with the lines of constant velocity is given by, using (2.34),

$$\tan \Gamma = \rho \cot z \qquad (2.36)$$

so that (2.35) may be rewritten,

$$\Theta = \rho^{-1}z + \Gamma + (2 - \rho^{-1})\frac{\pi}{2} \qquad (2.37)$$

(2.33) becomes

$$qt + x(q \cos \Theta)_q + y(q \sin \Theta)_q = 0$$

or, on performing the differentiation, using $q\Theta_q = \cot \Gamma$; and expressing x,y in terms of polar coordinates $(r,\varphi)$ $(x = r \cos \varphi,$ $y = r \sin \varphi)$,

$$qt \sin \Gamma = r \sin (\Theta - \Gamma - \varphi)$$

Finally, in terms of z, using (2.36), and (2.37),

$$R \sin (\rho^{-1}z - \varphi - \frac{\pi}{2\rho}) = \cos z \qquad (2.38)$$

where $R = r/\rho bt$. For any fixed z, equation (2.38) represents the $(R,\varphi)$ equation of a straight line and the envelope of the system as z varies will be given additionally by partial differentiation with respect to z; yielding a second line,

$$R \cos (\rho^{-1}z - \varphi - \frac{\pi}{2\rho}) = - \rho \sin z \qquad (2.39)$$

and therefore the equations in parametric form may be expressed in terms of z (in terms of which $\Gamma$, q, $\Theta$ are already known),

$$\varphi = \rho^{-1}z - \tan^{-1} (\rho \tan z) - \frac{\pi}{2\rho} + \frac{3\pi}{2} \qquad (2.40)$$

$$R^2 = \cos^2 z + \rho^2 \sin^2 z \qquad (2.41)$$

therefore the $(R,\varphi)$ equation may be explicitly written down if desired, since

$$\tan z = (\frac{1 - R^2}{R^2 - \rho^2})^{1/2}$$

and $0 < \rho \leq R \leq 1$ $\quad (\rho^2 = \frac{1}{11}$ for $\gamma = \frac{7}{5})$.

If in (2.40), z is regarded as the radial distance in polar coordinates $(z,\varphi)$ then (2.40) is the equation of an epicycloid, and since (2.41) may be rewritten

$$R^2 = (1 - \rho^2) \cos^2 z + \rho^2$$

the actual curve in the $(R,\varphi)$ may be drawn.

The envelope is similar to an epicycloid which increases linearly in size since $r = R\,\rho bt$. Along each tangent line the velocity vector is instantaneously constant and is inclined to the line at the angle $\Gamma$ according to (2.36). The instantaneous streamlines are perpendicular to the tangent generated from the point of the envelope given by $z = 0$ and then sweep round to approach asymptotically the direction parallel to the tangent at $z = \pi/2$.

The parameter line $z = 0$ along which the flow commences

moves normal to itself with velocity $\rho b$, equal to that of the fluid along it. The $z = 0$ line is perpendicular to the radius $\theta = -\beta$ where, $\beta = -\pi/2(\rho^{-1}-3)$ lies in the fourth quadrant, $(0 < \beta < \pi/2$ for $\gamma = 7/5$; $\rho^2 = 1/11)$.

3. Solution Involving an Initial Advancing Discontinuity.

When the $\lambda$-lines are parallel the flow is given by a solution $F(z;t)$ of the non-linear partial differential equation

$$F_{zt}^2 - F_{zz} F_{tt} + t F_{tt} + (\gamma - 1)F_t = 0 \tag{3.1}$$

the velocity parameter $z(x,t)$ being then given by the field equation

$$x = zt - F_z(z;t) \tag{3.2}$$

For the acoustic speed; $a^2 = \frac{\gamma - 1}{2}(c^2 - q_o^2 - z^2 - 2\Phi_t)$

that is $$a^2 = -(\gamma - 1)F_t(z;t)$$

so that for non-negative pressures we must consider solutions of (3.1) for which the time derivative is negative only.

An interesting solution exhibiting an unsteady singularity in two-dimensional flow may be obtained by separating the variables and trying

$$F(z;t) = Z(z)T(t)$$

giving as a possible solution,

$$tT_{tt} + (\gamma - 1)T_t = 0$$

provided

$$\frac{ZZ_{zz}}{Z_z^2} = \frac{T_t^2}{TT_{tt}} = \text{constant, independent of } z \text{ or } t.$$

The first implies $T(t) = A_1 t^{2-\gamma}$ (in terms of a constant A) and this is automatically compatible with the second and third conditions when

$$Z(z) = A_2 z^{\gamma-1}$$

so that we may take as a special solution of (3.1),

$$F(z;t) = -\frac{b}{(\gamma-1)(2-\gamma)} z^{\gamma-1} t^{2-\gamma} \qquad (3.3)$$

by suitable choice of the constant b which must be positive for $1 < \gamma < 2$. Taking $\gamma = 7/5$ then the parameter $z(x,t)$ is given implicitly by

$$x = zt + \frac{5b}{3}\left(\frac{t}{z}\right)^{3/5} \qquad (3.4)$$

Confining attention to $\left\{\begin{matrix} z \geq 0 \\ t \geq 0 \end{matrix}\right\}$; this equation defines z real only when x exceeds its value when

$$z = z_1(t) = \frac{b^{5/8}}{t^{1/4}}$$

at which

$$\frac{\partial x}{\partial z} = 0$$

which implies a concentration of differing z lines (wave lines) and rapid change in velocity at the station

$$x = \frac{8}{3} b^{5/8} t^{3/4}$$

$$= x_1(t) \text{ say.}$$

The wave line specified by constant z advances with speed

$$\frac{\partial x}{\partial t} = z + bz^{-3/5} t^{-2/5}$$

which is very large near $t = 0$ and decreases to a constant value z, equal to the x-component of the instantaneously local fluid

speed, as $t \to \infty$. In contrast, the singular line, which at $t = 0$ is crossing the y-axis with $\infty$ velocity, decelerates with speed eventually falling to zero as $t \to \infty$. This speed is

$$\frac{d}{dt}\left(\frac{8}{3} b^{5/8} t^{3/4}\right) = 2b^{5/8} t^{-1/4} = 2z_1(t)$$

is twice the x-component of the local fluid located on the line at time t. The instantaneous streamlines are defined by $dy/dx = q_o/z$ so that, regarding t as constant,

$$\frac{dy}{dz} = \frac{q_o}{z}\frac{dx}{dz} = \frac{q_o}{z}\left(t - \frac{bt^{3/5}}{z^{8/5}}\right)$$

therefore

$$(y - y_o)/q_o = t \log z + \frac{5b}{8} t^{3/5} z^{-8/5}$$

and

$$x = tz + \frac{5b}{3} t^{3/5} z^{-3/5}$$

may be regarded as the parametric equations of the streamline in terms of z as parameter, $y_o$ being a constant determining the different streamlines. Singular points in all such streamlines will occur when $\partial y/\partial z = \partial x/\partial z = 0$; but the curvature is infinite and $dy/dx$ remains finite and one valued. This occurs when $z = z_1(t)$ and it is seen that no streamline may be continued beyond the singular line into the region $x \leq x_1(t)$, corresponding to which no real values of $z(x,t)$ exist.

Also both branches of the x-z curve, corresponding to the variations

$$0 < z \leq z_1(t)$$

$$z_1(t) \leq z < \infty,$$

cannot be simultaneously considered at a given positive time t, because this would imply intersections of the streamlines and a double valued velocity at the point (x,y,t).

However, in unsteady motion the instantaneous streamlines do not yield a picture of the actual flow pattern and it is necessary to obtain explicitly the equations of the paths of the fluid elements. These are obtained, theoretically, by integrating the equations,

$$\frac{dx}{dt} = z(x,t); \qquad \frac{dy}{dt} = q_o$$

the parameter z being given by (3.4). It is necessary to adopt an alternative procedure and to eventually express x and y in terms of z, not the time. Form (3.4) by taking the total differential

$$dx = \frac{\partial x}{\partial t}\, dt + \frac{\partial x}{\partial z}\, dz$$

therefore the relation between z and t along a particle path is given by the ordinary differential equation, for z(t);

$$z = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial z}\frac{dz}{dt}$$

that is, using (3.4) explicitly;

$$\frac{dt}{dz} = \frac{t}{z} - \frac{t^{7/5} z^{3/5}}{b}$$

giving an integration in terms of a constant $\alpha$, depending on the element,

$$z^2 + \alpha_o = 5b\left(\frac{z}{t}\right)^{2/5} \qquad (3.5)$$

Therefore on substitution for t as a function of z;

$$\frac{3x}{(5b)^{5/2}} = \frac{4z^2 + \alpha}{(z^2 + \alpha)^{5/2}} ; \quad \frac{y - y_o}{q_o(5b)^{5/2}} = \frac{z}{(z^2 + \alpha)^{5/2}} \tag{3.6}$$

are the equations of the particle paths in which $(\alpha, q_o)$ determine the $\infty^2$ elements and $z$, the $x$ component of the fluid speed is used as fundamental parameter. The general form of the fluid element paths, according to (3.6), will differ according as $\alpha$ is positive or negative.

It is seen that when $\alpha > 0$ the fluid paths possess a cusp at $z = 1/2\sqrt{\alpha}$ . Then the element $(\alpha, q_o)$ will generate this cusp at time $t = 16\, b^{5/2}/\alpha^2$ along the line $x = 64/3\, b^{5/2}\alpha^{-3/2}$ that is when it meets the line $x = 8/3\, b^{5/8}t^{3/4}$ i.e. $x = x_1(t)$. When this occurs, the local acoustic speed, is given by

$$a^2 = b\left(\frac{z}{t}\right)^{2/5} = \frac{\alpha}{4} \tag{3.7}$$

so that $|a| = \frac{1}{2}\sqrt{\alpha} = z_1(t)$. Therefore, on the singular line, the streamlines intersect this line at the local Mach angle $\mu = \sin^{-1}\left|\frac{a}{q}\right|$ ; or the normal component of the fluid speed is sonic. In particular it follows that the singular line only occurs when the local fluid speed $(q_o^2 + z^2) > a^2$ is supersonic. The acceleration

$$z_t + \underline{q} \cdot \nabla z = z_t + z z_x = bz(z^{8/5}t^{7/5} - bt)^{-1}$$

becomes infinite when $z = b^{5/8}t^{-1/4} = z_1(t)$ so that, in effect, the solution breaks down on and to the left of the line $x = x_1(t)$. Any attempt to continue the flow into the region $0 \leq x < x_1(t)$, which was previously occupied by fluid in continuous motion,

must fail because the field Eqn. (3.4) will not give a real velocity for $x < x_1(t)$. The parametric equations (3.6) are similarly not applicable, and cannot yield real z for real $x < x_1(t)$.

<u>First Solution</u>. When elements $\alpha > 0$ are considered there are two possible paths. The first commencing at $t = 0$; $x = 4/3\ (5b)^{5/2}\alpha^{-3/2}$ continues to move to the right until it is eventually overtaken at time $t = 16\ b^{5/2}\alpha^{-2}$ by the singular line $x = x_1(t)$ which by that time has reached the station $x = 64/3\ b^{5/2}\alpha^{-3/2}[= x_1(t)]$. The relevant range for z is $0 \leq z \leq 1/2\sqrt{\alpha}$. When $\alpha \leq 0$, the corresponding fluid elements are concentrated along the axis $x = 0$ at zero time. This difficulty is avoided by confining attention to $\alpha > 0$ for which at $t = 0$ the instantaneous streamlines occupy the whole half plane $x \geq 0$ and comprise lines parallel to the $x = 0$ axis since $z = 0$. The singular line initially found along $x = 0$, moving at high speed, may be regarded as the result of a uniform impulsive pressure applied over this face (the $x = 0$ plane in three-dimensional flow) from which compression waves are generated. These waves proceed outwards with ever decreasing velocity, deflecting the fluid particles from their initially vertical paths.

The compression waves precede the advancing singular line $x = x_1(t)$ which overtakes the fluid instantaneously moving where x-component of velocity is locally sonic. At the instant of interception this component is abruptly doubled to equal the velocity $2\ z_1(t)$ of propagation of the discontinuity, the fluid being carried along in an ever increasing concentration (density)

along the singular line. This process causes the speed of propagation to reduce continuously and fall to zero as time increases. It should be noted that the advancing singular line cannot be considered, as a shock wave; the fluid overtaken has its speed discontinously increased but the normal velocity relative to that of the front rereduces to zero so that this change does not resemble passage through a shock. In addition the fluid does not pass through the singularity but accumulates along the front and is carried along with it, leaving a vacuum in the region behind.

Since $p \propto a^7$ and $a^2 = b(\frac{z}{t})^{2/5}$, the uniformly distributed surface pressure necessary to be applied along the boundary $x = x_1(t)$ is obtained when $z = z_1(t) = b^{5/8}t^{-1/4}$ therefore $p \propto t^{-1/4}$.

<u>Second Solution</u>. When the second branch $z_1(t) \leq z < \infty$ is considered flow obtained is quite different because at $t = 0$ there is an intense concentration (high density) of fluid along the $x = 0$ axis, across which it is instantaneously released with high speed, expanding into the adjacent vacuum $x > 0$ ($x < 0$ can be similarly considered). At a subsequent time $t$ the singular line has advanced to $x = x_1(t)$ with ever decreasing speed and the fluid released at $t = 0$ continues to precede the singular line also with ever decreasing speed. Unlike the phenomenon for $\alpha > 0$, the front never overtakes the preceding $\alpha < 0$ fluid again but falls back as time progesses. This may be demonstrated by focusing attention on conditions at a chosen time $t(> 0)$. At this time the front is moving forward with speed

$v = 2z_1(t) = 2b^{5/8}t^{-1/4}$ or $t = 2^4 b^{5/2} v^{-4}$, and the x-component of velocity of any particle $\alpha < 0$ is z given by $t=(5b)^{5/2}z(z^2+\alpha)^{-\frac{5}{2}}$ $\geq (5b)^{5/2}z^{-4}$. Therefore

$$2^4 b^{5/2} v^{-4} \geq (5b)^{5/2} z^{-4}$$

or

$$z \geq 5^{5/8} 2^{-1} v > v.$$

It follows, since this inequality is valid for all positive time, that all the $\alpha \leq 0$ fluid elements initially concentrated there precede the singular line. A similar argument shows that the $\alpha > 0$ elements are characterized by $z < v$, so that considerable information is obtained from the curve showing the variations of x with $\alpha$ $(-\infty < \alpha \leq 4b^{5/4}t^{-1/2})$ at any time $t > 0$.

This may be obtained by noting that $x(\alpha,t)$ is given jointly in terms of $(z,\alpha)$ by

$$\frac{3x}{(5b)^{5/2}} = \frac{4z^2 + \alpha}{(z^2 + \alpha)^{5/2}} \tag{3.8}$$

where $z(\alpha,t)$ is given by

$$\frac{t}{(5b)^{5/2}} = \frac{z}{(z^2 + \alpha)^{5/2}} \tag{3.9}$$

Ir follows by differentiation partially with respect to $\alpha$ (t constant) for $x(\alpha,t)$;

$$3z^2 \frac{\partial x}{\partial \alpha} = t(4z^2 - \alpha)\frac{\partial z}{\partial \alpha} + zt$$

$$2(4z^2 - \alpha)\frac{\partial z}{\partial \alpha} + 5z = 0$$

Therefore, eliminating $\partial z/\partial \alpha$,

$$\frac{\partial x}{\partial \alpha} = - \frac{t}{2z} \tag{3.10}$$

The subsidiary curves giving the variation of $x, \alpha$ with $z$ are first drawn and then combined to give, with the aid of (3.10), the curve $x \sim \alpha$. These curves have equations

$$x = zt + \frac{5b}{3} \left(\frac{t}{z}\right)^{8/5}; \qquad \alpha = 5b \left(\frac{z}{t}\right)^{2/5} - z^2$$

and generate a cusp at $[x_1(t), \alpha_1(t)]$ in the $(x,\alpha)$ locus where

$$x_1(t) = \frac{8}{3} b^{5/8} t^{3/4}; \qquad \alpha_1(t) = 4b^{5/4} t^{-1/2}$$

$$z_o(t) = (5b)^{5/8} t^{-1/4}; \qquad z_1(t) = b^{5/8} t^{-1/4}$$

The curve varies slightly with $t$ but it is seen how only one branch may be considered at a time, since for any given $(x,t)$ two different values of $\alpha$, with corresponding different velocities cannot be simultaneously considered. As $t$ increases the cusp $\alpha = 4b^{5/4} t^{-1/4}$ approaches zero, the total variation of $\alpha$ in the first solution $[0 < \alpha \leq 4b^{5/4} t^{-1/2};\ 0 < z \leq z_1(t)]$ being continuously reduced as more and more elements are caught in the advancing front. A similar state of affairs exists when the second solution $[-\infty < \alpha \leq 4b^{5/4} t^{-1/2};\ z_1(t) \leq z < +\infty]$ is considered, particles for which $\alpha > 0$ being removed from the regime of continuous fluid flow by the advancing front, but the $\alpha \leq 0$ elements are never reached and continue their motion uninterrupted for all time.

The variation of speed with time specified by (3.9) of elements for any $\alpha$; $-\infty < \alpha < +\infty$ relative to that of the singular line which is $z = 2z_1(t) = 2b^{5/8} t^{-1/4}$ shows that

1. Elements $\alpha < 0$ always move faster than the singular line at any time t.
2. Elements $\alpha > 0$ in range $\frac{1}{2}\sqrt{\alpha} \leq z < \infty$ initially move faster but are eventually overtaken by the singular line

These results follow because the intersection of the singular line curve A with the $\alpha > 0$ curve B occurs in the range

$$\frac{1}{2}\ |\alpha|^{1/2} \leq z \leq |\alpha|^{1/2}$$

The front of the disturbance which expands into the vacuum is therefore composed of high speed $\alpha < 0$ elements.

The field equation curve is shown in Fig. 1. The Figures 3, 4, 5, and 6 are used to draw the particle paths and Fig. 2 the instantaneous streamlines of the first solution. The corresponding curves for the particle paths of the second solution are shown in Figures 7, 8, 9, and 10.

4. Generalization of the Unsteady Simple Wave. When the direct extension of the simple wave $\underline{q}(\lambda)$ was considered in Section 2, the combined single equation for the velocity (obtained by simple elimination of $a^2$ between (2.1) and (2.2)) yielded the two equations (2.16), (2.17) which could be replaced by an ordinary differential equation for the velocity together with a partial differential equation for $\Phi(\lambda,t)$. There were two distinct possibilities but both were restrictive. Either the $\lambda$-lines were parallel (in which case the flows could be treated by a more direct method), or, in the case of a non-degenerate envelope, the time variation of $\Phi(\lambda,t)$ was linear. If the preceding method is adapted to flows of the type $\underline{q}(\lambda,t)$, in which the time

variation occurs independently as well as through the parameter $\lambda(x,y,t)$, by replacing $(x,y,t)$ by $(\lambda,\varphi,t)$ as fundamental independent variables, then the original single partial differential equation for the velocity regarded as a function of the three independent variables $(x,y,t)$ may be reduced to three partial differential equations for q, $\Theta$, $\Phi$ regarded as functions of the two variables $(\lambda,t)$. As in the simpler one parameter velocity field, this reduction is achieved by repeating the two fold transformation of $(x,y,t)$ into $(\lambda,\varphi,t)$, simultaneously using the Legendre contact transformation to transform the dependent function $\varphi$ into $\Phi$, whilst retaining $\varphi$ as a fundamental independent variable.

The reduction achieved is again due to the fact that $\Phi(x,y,t)$ reduces to a function of $(\lambda,t)$ only and is independent of $\varphi$. Such flows for which the velocity vector, reducing to a function of two independent variables $(\lambda,t)$ are sometimes referred to as "double waves" but the title simple wave is retained for the flows here discussed because the hodograph in the velocity plane still reduces to a curve, though this curve may vary with the time and generate a surface in the velocity time space.

The original analysis may be repeated by remembering that since $\underline{q} = \underline{q}(\lambda,t)$,

$$\frac{\partial \underline{q}}{\partial t} = \underline{q}_\lambda \frac{\partial \lambda}{\partial t} + \underline{q}_t$$

in which $\underline{q}_t$ does not mean the total time variation of $\underline{q}(\lambda,t)$ but simply the partial derivative with respect to time when $\lambda$ is constant. In the following, therefore, the suffix t is used to denote partial differentiation of a function of the type $f(\lambda,t)$

with respect to time when $\lambda$ is constant and not used to refer to the total differentiation with respect to time for which (x,y)are constant, except for $\varphi$ and $\lambda$ which are regarded as functions of (x,y,t). The following revised equations are obtained by successive differentiation

$$\Phi + \underline{r} \cdot \underline{q} = \varphi \tag{4.1}$$

$$\Phi_\lambda + \underline{r} \cdot \underline{q}_\lambda = 0 \tag{4.2}$$

$$\varphi_t = \Phi_t + \underline{r} \cdot \underline{q}_t \tag{4.3}$$

$$\varphi_{tt} = \Phi_{tt} + \underline{r} \cdot \underline{q}_{tt} + (\Phi_{\lambda t} + \underline{r} \cdot \underline{q}_{\lambda t})\lambda_t \tag{4.4}$$

$$(\Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda})\nabla\lambda + \underline{q}_\lambda = 0$$

$$(\Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda})\lambda_t + (\Phi_{\lambda t} + \underline{r} \cdot \underline{q}_{\lambda t}) = 0$$

whilst the equations (2.10) for the velocity must be replaced by

$$(a^2\underline{q}_\lambda - qq_\lambda \underline{q}) \cdot \nabla\lambda = 2q(q_\lambda\lambda_t + q_t) + \ddot{\varphi}$$

$$\frac{q^2}{2} + \frac{a^2}{\gamma - 1} + \Phi_t + \underline{r} \cdot \underline{q}_t = \frac{c^2}{2} \tag{4.5}$$

which becomes after substitution for $\varphi_{tt}$ and $\nabla\lambda$, $\lambda_t$,

$$\begin{aligned}(a^2\underline{q}_\lambda - qq_\lambda \underline{q}) \cdot \underline{q}_\lambda = {} & (2qq_\lambda + \Phi_{\lambda t} + \underline{r} \cdot \underline{q}_{\lambda t})(\Phi_{\lambda t} + \underline{r} \cdot \underline{q}_{\lambda t}) \\ & - (2qq_t + \Phi_{tt} + \underline{r} \cdot \underline{q}_{tt})(\Phi_{\lambda\lambda} + \underline{r} \cdot \underline{q}_{\lambda\lambda})\end{aligned} \tag{4.6}$$

It is noted that $\varphi_t$, $\varphi_{tt}$, and $a^2$ must continue to be regarded as general functions of position and time and the object is now to find their equivalent representations as functions of $(\lambda,\varphi,t)$.

The required expressions analogous to (2.13) are

$$\left.\begin{aligned}
\underline{q}_{\lambda\lambda} + A_1(\lambda,t)\underline{q} + B_1(\lambda,t)\underline{q}_\lambda &= 0\\
\underline{q}_{\lambda t} + A_2(\lambda,t)\underline{q} + B_2(\lambda,t)\underline{q}_\lambda &= 0\\
\underline{q}_{tt} + A_3(\lambda,t)\underline{q} + B_3(\lambda,t)\underline{q}_\lambda &= 0\\
\underline{q}_{t} + A_4(\lambda,t)\underline{q} + B_4(\lambda,t)\underline{q}_\lambda &= 0
\end{aligned}\right\} \qquad (4.7)$$

where the coefficients $A_i$, $B_i$ are functions of $(\lambda,t)$ only and are given by relations of the type

$$A_1\, \underline{q}_\Lambda \underline{q}_\lambda = \underline{q}_{\lambda\Lambda}\, \underline{q}_{\lambda\lambda}$$

$$B_1\, \underline{q}_\Lambda \underline{q}_\lambda = -\, \underline{q}_\Lambda \underline{q}_{\lambda\lambda}$$

therefore explicitly

$$\left.\begin{aligned}
A_1 &= -\frac{q^3\theta_\lambda^2}{2q_\lambda} \cdot \frac{\partial}{\partial\lambda}\left[q_\lambda^2 q^{-4}\theta_\lambda^{-2} + q^{-2}\right]\\
B_1 &= -\frac{\partial}{\partial\lambda}(\log q^2\theta_\lambda)\\
A_1 &= \frac{q_\lambda^2 + q^2\theta_\lambda^2}{q^2\theta_\lambda}\, \frac{\partial}{\partial\lambda}\left[\theta + \tan^{-1}\frac{q\theta_\lambda}{q_\lambda}\right]\\
A_2 &= \frac{q_\lambda^2 + q^2\theta_\lambda^2}{q^2\theta_\lambda}\, \frac{\partial}{\partial t}\left[\theta + \tan^{-1}\frac{q\theta_\lambda}{q_\lambda}\right] \;; \qquad B_2 = -\frac{(q\theta_\lambda)_t + q_\lambda\theta_t}{q\theta_\lambda}\\
A_3 &= \frac{q_\lambda}{q\theta_\lambda}\,\theta_{tt} + \theta_t^2 - \frac{q_{tt}}{q} + \frac{2q_t\theta_t q_\lambda}{q^2\theta_\lambda} \;; \qquad B_3 = \frac{-1}{q^2\theta_\lambda}\,\frac{\partial}{\partial t}(q^2\theta_t)\\
A_4 &= \frac{q_\lambda\theta_t - q_t\theta_\lambda}{q\theta_\lambda} \;; \qquad B_4 = -\frac{\theta_t}{\theta_\lambda}
\end{aligned}\right\} \qquad (4.8)$$

The actual computation of these coefficients is best done as follows. With the usual notation for $\underline{t}(\Theta) = (\cos\Theta, \sin\Theta)$ the unit tangent vector along the streamline,

$$\underline{q} = q\underline{t}(\Theta)$$

therefore

$$\underline{q}_\lambda = q_\lambda \underline{t} + q\Theta_\lambda \underline{n}$$

$$\underline{q}_{\lambda\lambda} = (q_{\lambda\lambda} - q\Theta_\lambda^2)\underline{t} + (2q_\lambda\Theta_\lambda + q\Theta_{\lambda\lambda})\underline{n}$$

therefore

$$q^2\Theta_\lambda \underline{q}_{\lambda\lambda} + [q(q_\lambda\Theta_{\lambda\lambda} - \Theta_\lambda q_{\lambda\lambda}) + 2q_\lambda^2\Theta + q^2\Theta_\lambda^3]\underline{q}$$

$$= (2q_\lambda\Theta_\lambda + q\Theta_{\lambda\lambda})q\underline{q}_\lambda$$

or

$$\underline{q}_{\lambda\lambda} + A_1(\lambda,t)\underline{q} + B_1(\lambda;t)\underline{q}_\lambda = 0$$

where

$$q_\lambda^2\Theta A_1(\lambda,t) = (q_\lambda^2 + q^2\Theta_\lambda{}^2)\Theta_\lambda + q^2\left(\frac{q\Theta_\lambda}{q_\lambda}\right)_\lambda$$

after some rearrangement,

$$q^2\Theta_\lambda\, A_1(\lambda,t) = (q_\lambda^2 + q^2\Theta_\lambda^2)\,\frac{\partial}{\partial\lambda}\left[\Theta + \tan^{-1}\frac{q\Theta_\lambda}{q_\lambda}\right]$$

or, equivalently,

$$A_1(\lambda;t) = -\frac{q^3\Theta_\lambda^2}{2q_\lambda}\,\frac{\partial}{\partial\lambda}\left[(q^{-2} + \frac{q_\lambda^2}{q^4\Theta_\lambda^2}\right]$$

Similar procedures yield the explicit expressions for the remaining coefficients.

Scalar multiplication of equations (4.7) by $\underline{r}$ and then using (4.1) and (4.2) to substitute for the scalar products $\underline{r} \cdot \underline{q}$ ; $\underline{r} \cdot \underline{q}_\lambda$ gives

$$\left.\begin{aligned} \underline{r} \cdot \underline{q}_{\lambda\lambda} &= (\Phi - \varphi)A_1 + \Phi_\lambda B_1 \\ \underline{r} \cdot \underline{q}_{\lambda t} &= (\Phi - \varphi)A_2 + \Phi_\lambda B_2 \\ \underline{r} \cdot \underline{q}_{tt} &= (\Phi - \varphi)A_3 + \Phi_\lambda B_3 \\ \underline{r} \cdot \underline{q}_{t} &= (\Phi - \varphi)A_4 + \Phi_\lambda B_4 \end{aligned}\right\} \tag{4.9}$$

and direct substitution into (4.6) and (4.5) gives

$$X(\lambda,t) + (\Phi - \varphi)Y(\lambda,t) + (\Phi - \varphi)^2 Z(\lambda,t) = 0 \tag{4.10}$$

in which the functions X,Y,Z are obtained as functions of $\underline{q}(\lambda,t)$, $\Phi(\lambda,t)$ only and are therefore independent of $\varphi$. It follows that if the field equation (4.10) is to be valid everywhere then all the coefficients must be illusory, that is

$$X(\lambda,t) = Y(\lambda,t) = Z(\lambda,t) = 0 \tag{4.10a}$$

which are the three required partial differential equations for q, $\Theta$, $\Phi$ in terms of the independent variables $(\lambda,t)$.

Explicitly

$$Z(\lambda,t) = A_2^2 - A_1A_3 \tag{4.11}$$

which is independent of $\Phi(\lambda,t)$.

$$X(\lambda,t) = (\Phi_{\lambda t} + 2qq_{\lambda\lambda} + B_2\Phi_\lambda)(\Phi_{\lambda t} + B_2\Phi_\lambda)$$
$$- (\Phi_{tt} + 2qq_t + B_3\Phi_\lambda)(\Phi_{\lambda\lambda} + B_1\Phi_\lambda) - (\gamma - 1)\left\{\frac{c^2 - q^2}{2}\right.$$
$$\left. - \Phi_t - B_4\Phi_\lambda\right\} \underline{q}_\lambda^2 + q^2 q_\lambda^2 \quad (4.12)$$

$$Y(\lambda,t) = 2A_2(\Phi_{\lambda t} + qq_\lambda + B_2\Phi_\lambda) - A_3(\Phi_{\lambda\lambda} + B_1\Phi_\lambda)$$
$$- A_1 (\Phi_{tt} + 2qq_t + B_3\Phi_\lambda) + (\gamma - 1)A_4 \underline{q}_\lambda^2 \quad (4.13)$$

In general, solutions of the equations (4.10a), (4.11), (4.12), and (4.13) are very difficult to obtain, such flows corresponding to the general double wave solution. The original simplified theory of the unsteady simple wave may be extracted by noting that in that case all the derivatives $\underline{q}_t$, $\underline{q}_{\lambda t}$, $\underline{q}_{tt}$ vanish therefore $A_2 = A_3 = A_4 = B_2 = B_3 = B_4 = 0$ so that

$$X(\lambda,t) = (qq_\lambda + \Phi_{\lambda t})^2 - \Phi_{tt}(\Phi_{\lambda\lambda} + B_1\Phi_\lambda) - a^2\underline{q}_\lambda^2$$

$$Y(\lambda,t) = - A_1(\lambda)\Phi_{tt}$$

$$Z(\lambda,t) = 0$$

The condition $Z = 0$ is therefore automatically satisfied and the additional relations $X = Y = 0$ reproduce the equations (2.16), and (2.17).

<u>The λ-Envelope</u>. When the generalization of the simple wave to steady rotational compressible flow it is found that the λ-lines are concurrent or parallel (which may be regarded as a particular case of concurrent lines when the point of concurrency

recedes to infinity) and that in no case is a non-degenerate envelope possible. This is in contrast to the original steady irrotational simple wave, for which an arbitrary curve may be assigned in advance to be the envelope of the $\lambda$-lines from which the flow is constructed. In rotational and unsteady incompressible flow it is also found that there is no non-degenerate envelope. The simple wave in unsteady flow however represents a much more general class of flows because it is possible to obtain solutions for which the $\lambda$-lines are neither parallel nor concurrent. The envelope E(t) would be obtained, theoretically, by solving for $\underline{r}$ the equations

$$\left.\begin{aligned} \Phi_\lambda + \underline{r} \cdot \underline{q}_\lambda &= 0 \\ \Phi_{\lambda\lambda} + r \cdot \underline{q}_{\lambda\lambda} &= 0 \end{aligned}\right\} \qquad (4.14)$$

This would be possible provided $\underline{q}_{\lambda\lambda}\underline{q}_{\lambda\lambda} \neq 0$ (i.e. $A_1 \neq 0$), the exceptional case occurring when the $\lambda$-lines are all parallel. If the lines are concurrent, then the point of concurrency may vary with the time as would the general envelope vary with the time. If however they are concurrent through a fixed point, this point may be selected to coincide with the origin without loss of generality and then $\Phi_\lambda(\lambda,t) = 0$ so that $\Phi$ reduces to a function of time only.

To construct the flow it is necessary to draw the envelope which is given in parametric form $\underline{r}(\lambda,t)$ from equations (4.14). On the envelope there is a relation between $\lambda$ and $\varphi$ given by

$$\Phi_{\lambda\lambda} + (\Phi - \varphi)A_1 + \Phi_\lambda B_1 = 0 \qquad (4.15)$$

in virtue of (4.14) and (4.9), it follows that $\nabla\lambda$ and $\lambda_t$ become infinite on E(t) and that the acceleration

$$\frac{\partial}{\partial t}\underline{q}\,(x,y,t) + (\underline{q}\cdot\nabla)\underline{q} = (\lambda_t + \underline{q}\cdot\nabla\lambda)\underline{q}_\lambda + \underline{q}_t$$

also becomes infinite. The solution cannot be continued as far as the envelope where the λ-lines coalesce and the velocity is ambiguous. However the flow field of any solution is obtained automatically by drawing the λ-lines generated as the tangents to the envelope. Along any such λ-line, the value of λ and therefore of $\underline{q}(\lambda,t)$ is constant and known from (4.14).

The parametric representation of the plane is obtained from (4.1), and (4.2),

$$\underline{r}(\lambda,\varphi,t) = [(\Phi - \varphi)\underline{Q}_\lambda - \Phi_\lambda\underline{Q}]\exp\int B_1(\lambda,t)d\lambda \qquad (4.16)$$

since $q^2\theta_\lambda = \exp - \int B_1(\lambda,t)d\lambda$, and $\underline{Q} = \underline{K}_\Lambda\underline{q}$. By direct differentiation

$$\underline{r}_\lambda = -[\Phi_{\lambda\lambda} + B_1\Phi_\lambda + A_1(\Phi - \varphi)]\,\underline{Q}\exp\int B_1(\lambda,t)d\lambda$$

$$\underline{r}_\varphi = -\underline{Q}_\lambda\exp\int B_1(\lambda,t)d\lambda$$

The relation between λ and φ along an instantaneous streamline will be obtained by integrating the differential definition $\underline{q}_\Lambda . d\underline{r} = 0$ where

$$d\underline{r} = \underline{r}_\lambda\,d\lambda + \underline{r}_\varphi\,d\varphi$$

therefore

$$[\Phi_{\lambda\lambda} + B_1\Phi_\lambda + A_1(\Phi - \varphi)]q\,d\lambda + q_\lambda\,d\varphi = 0$$

In unsteady flow however it is the actual paths of the fluid elements which show the motion. These paths are obtained by

integrating the differential equations

$$\frac{d}{dt}\,\underline{r}(\lambda,\varphi,t) = \underline{q}(\lambda,t)$$

that is

$$\underline{r}_\lambda \frac{d\lambda}{dt} + \underline{r}_\varphi \frac{d\varphi}{dt} + \underline{r}_t = \underline{q}(\lambda,t) \tag{4.17}$$

from which are obtained two ordinary simultaneous differential equations for $\lambda$ and $\varphi$ regarded as functions of t only.

On integration in terms of two constants $(c_1,c_2)$, which serve to determine the initial position of the $\infty^2$ elements, $\lambda$ and $\varphi$ are obtained as functions of time which when substituted in (4.16) yield the position vector of the element $(c_1,c_2)$ in the form $\underline{r}(t,c_1,c_2)$.

Singularities. An inherent disadvantage of inverse methods such as the one here employed is that it is difficult to avoid in advance solutions of the final equations giving rise to discontinuities in the streamlines in the physical plane. The velocity may be a many valued function of position corresponding to those regions in the physical plane for which the solution may not be mathematically unique, but it is possible to obtain a physically possible flow pattern by confining attention to just one branch of the $\lambda \sim \underline{r}$ curves. In the physical plane the instantaneous streamlines and particle paths corresponding to the different branches usually meet in a cusp, where the acceleration becomes infinite and the flow is supersonic. Each solution cannot be continued beyond the cusp locus (limit line) or even up to it because such singularities represent a local breakdown of the continuous flow theory on which the flow was constructed.

The fact that the acceleration becomes infinite where the local fluid speed is supersonic suggests that it should be possible to replace the limit line by a shock curve, however it is not usually possible to satisfy all the necessary shock relations along the limit line. In the simple one parameter field $\underline{q}(\lambda)$ of Sections 2, 3, this substitution was precluded because the limit lines intersected the streamlines at the local Mach angle, and on existing shock theory any shock inclined at the local Mach angle has vanishing intensity simply reducing to a Mach wave.

The Simple Centred Wave. The equations (4.11), (4.12), and (4.13) are so complex that it is desirable first to investigate the possible existence of flows for which the λ-lines are concurrent through a fixed point, and defer the general study of these equations until Section 5. There is also the additional possibility that the point of concurrency may vary with the time. In the simpler case, however, the origin may be selected as the fixed point through which all the λ-lines pass for all time, then $\Phi_\lambda = 0$ and $\Phi$ reduces to a function of time only. The three conditions X = Y = Z = 0 then give,

$$\left.\begin{aligned} (\gamma - 1)(c^2 - q^2 - 2\Phi_t)\,\underline{q}_\lambda^2 &= 2q^2q_\lambda^2 \\ 2A_2qq_\lambda + (\gamma - 1)A_4\underline{q}_\lambda^2 &= A_1(\Phi_{tt} + 2qq_t) \\ A_2^2 &= A_1A_3 \end{aligned}\right\} \qquad (4.18)$$

The functions $A_i$ are given directly in terms of $q(\lambda;t)$, $\theta(\lambda;t)$ and their derivatives by equations (4.8). $\Phi(t)$ may be eliminated between the first two equations of (4.18) to yield an equation for

$\underline{q}(\lambda;t)$ in addition to the third equation, $A_2^2 = A_1A_3$. This equation is

$$2A_2qq_\lambda + (\gamma - 1)A_4\,\underline{q}_\lambda^2 = A_1\left\{qq_t - \frac{1}{\gamma - 1}\frac{\partial}{\partial t}\left(\frac{q^2q_\lambda^2}{\underline{q}_\lambda^2}\right)\right\} \qquad (4.19)$$

(4.18), and (4.19) are therefore the general equations for the velocity for the unsteady simple wave when the "centre" is fixed. It is obtained in terms of any parameter $\lambda(x,y,t)$ (for which $\nabla\lambda$ does not vanish) which may be taken to be the velocity $q(x,y,t)$. The original functional relation $q = q(\lambda,t)$ now simply reduces to $q = \lambda$, so that $q_\lambda = 1$, $q_{\lambda\lambda} = q_t = q_{tt} = q_{\lambda t} = 0$, whilst $\theta(\lambda;t)$ becomes $\theta(q;t)$ and $\theta_\lambda = \theta_q$, $\theta_{\lambda\lambda} = \theta_{qq}$, etc.

Equation (4.19) now reduces to

$$2A_2q + (\gamma - 1)A_4\,\underline{q}_q^2 = -\frac{q^2A_1}{\gamma - 1}\frac{\partial}{\partial t}\left(\underline{q}_q^{-2}\right) \qquad (4.20)$$

where

$$\left.\begin{aligned}
\underline{q}_q^2 &= 1 + q^2\theta_q^2 \\
A_1 &= \frac{1 + q^2\theta_q^2}{q^2\theta_q}\frac{\partial}{\partial q}\left[\theta + \tan^{-1}q\theta_q\right] \\
A_2 &= \frac{1 + q^2\theta_q^2}{q^2\theta_q}\frac{\partial}{\partial t}\left[\theta + \tan^{-1}q\theta_q\right] \\
A_3 &= \theta_t^2 + \frac{\theta_{tt}}{q\theta_q}, \qquad A_4 = \frac{\theta_t}{q\theta_q}
\end{aligned}\right\} \qquad (4.21)$$

Equations (4.18) and (4.20) now become two partial differential equations for the single dependent function $\theta(q;t)$, $q$ now being regarded as an independent variable. Thus it appears that the flow is overdetermined. The above simplification of the general

theory may be obtained directly by resorting to the equations (2.1) and (2.2). It should be noted however that this direct approach is only possible when the λ-lines are <u>concurrent</u> through a <u>fixed</u> point, and may therefore be considered as radii through the origin when the current polar coordinate r (the radial distance) is of no consequence to the velocity $\underline{q}$. It is not capable of direct generalization to reproduce the more general equations (4.10a) for which there exists a time varying envelope. In view of the presence of three equations for just two functions $(\Phi, \Theta)$ or, in the simpler case, two equations for $\Theta$, these equations will be established in the simpler case, by a more direct method. Having thus verified their validity, it will be then justifiable to return to the general equations (4.11), (4.12), and (4.13).

<u>Direct Procedure of the Simplified Case</u>. Let u,v be the components of velocity along and normal to the radius vector drawn from the origin, $\sigma$ the radial angle and $\psi = \Theta - \sigma$ the angle $\underline{q}$ makes with the radius vector. Then $u = q \cos \psi$, $v = q \sin \psi$ and the general equation for the velocity,

$$a^2\left(u_r + \frac{u}{r} + \frac{1}{r} v_\sigma\right) = q\left(uq_r + \frac{v}{r} q_\sigma\right) + 2q\mathring{q} + \ddot{\varphi}$$

reduces to, since r is of no consequence to $\underline{q}(\sigma;t)$,

$$a^2(u + v_\sigma) = vqq_\sigma + 2rq\mathring{q} + r\ddot{\varphi}$$

that is,

$$(a^2 - q^2)q_\sigma \sin \psi + a^2 q\Theta_\sigma \cos \psi = 2rq\mathring{q} + r\ddot{\varphi} \qquad (4.22)$$

Now since $\underline{q}(\sigma,t)$ is independent of r

$$\varphi(r,\sigma;t) = \int \underline{q} \cdot d\underline{r} = \underline{q} \cdot \underline{r} + \Phi(t)$$

reduces to

$$\varphi = \Phi(t) + rq \cos \psi$$

Also the condition $\nabla_\wedge \underline{q} = 0$, i.e., $v = u_\sigma$ gives

$$\cot \psi = q\Theta_q \tag{4.23}$$

from which by differentiation is obtained

$$\dot{q}\Theta_q + q(\Theta_{qt} + \dot{q}\Theta_{qq}) = -\operatorname{cosec}^2 \psi \dot{\Theta}$$

regarding $\Theta(\sigma;t)$ as a function of $(q;t)$ so that $\dot{\Theta} = \Theta_q \dot{q} + \Theta_t$, etc. Therefore

$$\dot{q}[q\Theta_{qq} + (1 + \operatorname{cosec}^2\psi)\Theta_q] + q\Theta_{qt} + \Theta_t \operatorname{cosec}^2\psi = 0$$

Similarly

$$q_\sigma[q\Theta_{qq} + (1 + \operatorname{cosec}^2\psi)\Theta_q] = \operatorname{cosec}^2\psi$$

The expressions for the time derivatives of $\varphi$ are

$$\dot{\varphi} = \Phi_t - rq\Theta_t \sin \psi$$

$$\ddot{\varphi} = \Phi_{tt} + \frac{A(q;t)r \sin \psi}{q\Theta_{qq} + (1 + \operatorname{cosec}^2\psi)\Theta_q}$$

after successive differentiation and substitution in the resulting expressions of $\dot{q}$, where

$$A(q;t) = (\Theta_t \operatorname{cosec}^2\psi + q\Theta_{qt})^2 - q(q\Theta_{qq} + \overline{1+\operatorname{cosec}^2\psi}\cdot\Theta_q)(\Theta_{tt}+\Theta_t^2\cot\psi)$$

and the expression for the acoustic speed becomes

$$\frac{a^2}{\gamma - 1} = \frac{1}{2}(c^2 - q^2) - \Phi_t + rq\Theta_t \sin \psi$$

When these expressions are inserted in (4.22) and the coefficients of $(1 : r : r^2)$ equated to zero there results the three relations

$$\left.\begin{aligned}
(\gamma - 1)(c^2 - q^2 - 2\Phi_t) &= 2q^2\sin^2\psi \\
(\gamma-1)\theta_t \operatorname{cosec}^2\psi + 2(q\theta_{qt}+\theta_t \operatorname{cosec}^2\psi) &= q^{-1}\Phi_{tt}(q\theta_{qq}\overline{+1+\operatorname{cosec}^2\psi\cdot\theta_q}) \\
A(q;t) &= 0
\end{aligned}\right\} \qquad (4.24)$$

which may be reconciled with relations (4.18) by noting that, in (4.21), since $\sigma = \theta - \psi = \theta - \cot^{-1}q\theta_q = \theta + \tan^{-1}q\theta_q +$ constant, then

$$\left.\begin{aligned}
A_2 &= \frac{\operatorname{cosec}^2\psi}{q\cot\psi}\,\sigma_t = \frac{q\theta_{qt} + \theta_t \operatorname{cosec}^2\psi}{q^2\theta_q} \\
A_1 &= \frac{\operatorname{cosec}^2\psi}{q\cot\psi}\,\sigma_q = \frac{q\theta_{qq} + (1 + \operatorname{cosec}^2\psi)\theta_q}{q^2\theta_q} \\
A_3 &= \tan\psi(\theta_{tt} + \theta_t^2 \cot\psi) \qquad A_4 = \theta_t \tan\psi
\end{aligned}\right\} \qquad (4.25)$$

and

$$A(q;t) = (A_2^2 - A_1A_3)q^4\theta_q^2$$

Equations (4.24) therefore reduce to

$$\left.\begin{aligned}
(\gamma - 1)(c^2 - q^2 - 2\Phi_t)\operatorname{cosec}^2\psi &= 2q^2 \\
(\gamma - 1)A_4 \operatorname{cosec}^2\psi + 2qA_2 &= A_1 \Phi_{tt} \\
A_2^2 - A_1A_3 &= 0
\end{aligned}\right\} \qquad (4.26)$$

These coincide with (4.18) since, in the original notation, $q_t = 0$, and $\underline{g}_q^2 = 1 + q^2\theta_q^2 = \operatorname{cosec}^2\psi$ .

5. <u>The General Case</u>. The following analysis deals with the case when the parameter lines do not necessarily possess a degenerate envelope. The relations (4.21) are transformed to give, in terms of $p = \theta_z$; $q = \theta_t$

$$A_1 = \frac{1}{p}\left\{p(1 + p^2) + \theta_{zz}\right\}e^{-2z} \qquad A_2 = \frac{1}{p}\left\{q(1 + p^2) + \theta_{zt}\right\}e^{-z}$$

$$A_3 = \frac{1}{p}\left\{pq^2 + \theta_{tt}\right\} \qquad A_4 = \frac{q}{p}$$

since $z = \log q$; $q\theta_q = \theta_z = p$. Similarly the expressions for the functions $B_i$; (4.8) become,

$$-\,e^zB_1 = 1 + \frac{\theta_{zz}}{p} \qquad -\,e^{-z}B_3 = \frac{\theta_{tt}}{p}$$

$$-\,B_2 = \frac{q + \theta_{zt}}{p} \qquad -\,e^{-z}B_4 = \frac{q}{p}$$

where q now denotes the derivative $\theta_t$, and is not to be confused with the total fluid speed now denoted by $e^z$.

The angle $\sigma$ which the parameter lines make with a fixed direction $\theta = 0$ is given, apart from a constant additive angle, by

$$\sigma = \theta + \tan^{-1}\theta_z$$

that is

$$\sigma = \theta + \tan^{-1}p$$

since $\Gamma$, the angle at which the streamlines intersect the parameter lines, is given by

$$\cot\Gamma = \theta_z = p$$

and $\sigma = \theta - \Gamma$.

<u>Solution of the Velocity Equation</u>. Of the three fundamental simple wave equations (4.11), (4.12), and (4.13), only the last two involve $\Phi(q;t)$. The relation (4.11) involves only the velocity and is independent of $\Phi$, therefore it is possible to obtain general properties of the flow predicted by the common equation (4.11) for any possible form of $\Phi(q;t)$. (4.11) $(A_2^2 = A_1A_3)$ gives for the equation for $\Theta(z;t)$,

$$[\Theta_{zt} + q(1 + p^2)]^2 = [\Theta_{tt} + pq^2] \cdot [\Theta_{zz} + p(1 + p^2)]$$

which becomes

$$(\Theta_{zz}\Theta_{tt}-\Theta_{zt}^2) + pq^2\Theta_{zz}+p(1+p^2)\Theta_{tt}-2q(1+p^2)\Theta_{zt}-q^2(1+p^2)=0 \qquad (5.1)$$

which is equation of the Mange-Ampère type. It is non-linear but the second order terms of the second degree occur only as the Jacobian $\partial(p,q)/\partial(z,t) = \Theta_{zz}\Theta_{tt} - \Theta_{zt}^2$ therefore it can be dealt with by considering the characteristics which are given by (cf. Forsyth, Differential Equations: p. 485).

$$dp + p(1 + p^2)dz + \lambda dt = 0$$

$$dq + \lambda dz + pq^2 dt = 0$$

where $\lambda$ is given by the quadratic

$$\lambda^2 - 2\lambda q(1 + p^2) + q^2(1 + p^2)^2 = 0$$

that is $\lambda = q(1 + p^2)$. Thus the equation for $\Theta(z;t)$ is always parabolic, the roots $\lambda$ being real and coincident.

The coincident characteristics are given by

$$dp + p(1 + p^2)dz + q(1 + p^2)dt = 0$$

$$dq + q(1 + p^2)dz + pq^2dt = 0$$

which may be combined to eliminate dz in the form

$$qdp - pdq + q^2dt = 0$$

so that, if u is an arbitrary parameter of integration,

$$t + \frac{p}{q} = u \tag{5.2}$$

Similarly, elimination of dt yields,

$$pqdp - (1 + p^2)dq - q(1 + p^2)dz = 0$$

so that on integration in terms of another parameter v,

$$q^2e^{2z} = v(1 + p^2) \tag{5.3}$$

It follows that the "Intermediate Integral" of (5.1) may be obtained by taking any arbitrary function of (u,v), that is

$$F(u,v) = 0$$

which, being an equation (involving p,q) of the first order may be further integrated for $\Theta(x,t)$, if desired by Charpit's general method for such non-linear first order partial differential equations.

The general intermediate integral may be written

$v = f(u)$, $$q^2e^{2z} = (1 + p^2)f(t + \frac{p}{q}) \tag{5.4}$$

where f is arbitrary. However, it is possible to evade the

general method by solving (5.2), and (5.3) for p;q as functions of (z,t,u,v) giving, since $p = q(u - t)$,

$$q = \frac{\sqrt{v}}{\sqrt{e^{2z} - v(u - t)^2}} \; ;$$

$$p = \frac{\sqrt{v} \cdot (u - t)}{\sqrt{e^{2z} - v(u - t)^2}}$$

where actually $v = f(u)$, an arbitrary function of u.

Then insertion in

$$d\Theta = pdz + qdt$$

gives

$$\frac{d\Theta}{\sqrt{v}} = \frac{(u - t)dz + dt}{\sqrt{e^{2z} - v(u - t)^2}}$$

so that

$$\Theta - c = \cos^{-1}\left\{(u - t)\sqrt{v}\, e^{-z}\right\}$$

therefore $\Theta(z,t)$ is given by

$$\cos(\Theta - c) = (u - t)\sqrt{v}\, e^{-z}$$

or

$$q \cos(\Theta - c) = (u - t)\sqrt{v}$$

where q now denotes the fluid speed.

If c, v are regarded as arbitrary functions of a single parameter $\lambda (= u)$ then

$$q \cos\left\{\Theta - \varphi(\lambda)\right\} = \psi(\lambda)(\lambda - t) \qquad (5.5)$$

is the "complete" integral, from which the "general" integral is obtained by partial differentiation with respect to $\lambda$,

$$\varphi'(\lambda)q \sin\left\{\theta - \varphi(\lambda)\right\} = (\lambda - t)\psi'(\lambda) + \psi(\lambda) \qquad (5.6)$$

and eliminating $\lambda$ between (5.5), and (5.6) [definitely chosen functions $\varphi(\lambda)$, $\psi(\lambda)$ would be necessary in actually carrying out this procedure].

It follows from (5.5) in terms of $z = \log q$

$$\cos\left[\theta - \varphi(\lambda)\right] = \psi(\lambda)(\lambda - t)e^{-z} \qquad (5.7)$$

therefore partial differentiation with respect to z gives, after considering the corresponding variation of $\lambda(z,\theta,t)$ which according to (5.6) vanishes, giving

$$\theta_z = \cot\left[\theta - \varphi(\lambda)\right] \qquad (5.8)$$

therefore

$$\sigma = \theta - \cot^{-1}\theta_z$$

reduces to, simply

$$\sigma = \varphi(\lambda) \qquad (5.9)$$

(choosing principal value for convenience). $\sigma$ denotes the inclination of any local parameter line (along which $\theta$,q are instantaneously constant) to $\theta = 0$, therefore these lines will constitute a system of parallel lines if $\varphi(\lambda)$ reduces to a constant. Since in this case the left hand side of (5.6) vanishes and the right hand side then merely implies that t is an arbitrary function of $\lambda$, then $\lambda$ is an arbitrary function of t so that (5.5) gives

$$q \cos(\theta - \theta_0) = F(t)$$

F(t) an arbitrary function of t and $\theta_0$ a constant.

The combination of (5.5), and (5.6) is not the most

general solution for $\Theta(q;t)$; a more general solution may be obtained by applying Imschenetsky's method by inserting in (5.1) the solution

$$e^{+z}\cos(\Theta - c) = b(a - t)$$

and allowing the parameters (a,b) to vary, regarding c as a function of (a,b). However attention will be confined to the class of solutions obtained by eliminating $\lambda(q;t)$ between

$$q\cos(\Theta - \lambda) = \varphi(\lambda)[\psi(\lambda) - t] \tag{5.10}$$

$$q\sin(\Theta - \lambda) = \varphi'(\lambda)\{\psi(\lambda) - t\} + \varphi(\lambda)\psi'(\lambda) \tag{5.11}$$

which are obtained from (5.7), and (5.8) after a slight change of notation. With this notation, $\lambda$ becomes equal to $\sigma$ ( equal the inclination of any parameter line to $\Theta = 0$), and is defined implicitly in terms of the arbitrary functions $\varphi(\lambda)\cdot\psi(\lambda)$ by means of

$$q^2 = (\varphi^2 + \varphi_\lambda^2)(\psi - t)^2 + 2\varphi\varphi_\lambda\psi_\lambda(\psi - t) + \varphi^2\psi_\lambda^2 \tag{5.12}$$

obtained from (5.10), and (5.11) by elimination of $\Theta$. It is always possible to rewrite the system (5.5), (5.6) in the form (5.10), and (5.11) provided that, in the original system the function $\varphi(\lambda)$ [occuring in (5.5), and (5.6)] does not reduce to a constant.

Since

$$\left.\begin{aligned} q\Theta_q &= \cot\alpha = \Theta_z \\ q\Theta_t &= \varphi(\lambda)\operatorname{cosec}\alpha \\ \alpha &= \Theta - \lambda \\ \lambda(q;t) &= \Theta - \cot^{-1}q\Theta_q \end{aligned}\right\} \tag{5.13}$$

then by differentiation

$$\left.\begin{aligned} q\theta_{zz} &= -(q\cot\alpha - \lambda_q)\operatorname{cosec}^2\alpha \\ q^2\theta_{tt} &= -\varphi^2\operatorname{cosec}^2\alpha\cot\alpha + q\operatorname{cosec}\alpha(\varphi_\lambda + \varphi\cot\alpha)\lambda_t \\ q\theta_{zt} &= -\operatorname{cosec}^2\alpha\left\{\varphi\operatorname{cosec}\alpha - q\lambda_t\right\} \end{aligned}\right\} \qquad (5.14)$$

The partial derivatives of $\lambda(q;t)$ may be obtained from (5.12), giving

$$\left.\begin{aligned} L\Delta\lambda_t &= K \\ L\Delta\lambda_q &= q \end{aligned}\right\} \qquad (5.15)$$

where

$$\left.\begin{aligned} L &= \varphi_\lambda(\psi - t) + \varphi\psi_\lambda \\ K &= (\varphi^2 + \varphi_\lambda^2)(\psi - t) + \varphi\varphi_\lambda\psi_\lambda \\ \Delta &= (\varphi + \varphi_{\lambda\lambda})(\psi - t) + 2\varphi_\lambda\psi_\lambda + \varphi\psi_{\lambda\lambda} \end{aligned}\right\} \qquad (5.16)$$

so that (5.14) becomes, since,

$$\left.\begin{aligned} q\cos\alpha &= \varphi(\psi - t) \\ q\sin\alpha &= \varphi_\lambda(\psi - t) + \varphi\psi_\lambda = L \\ q^2 &= L^2 + \varphi^2(\psi - t)^2 \end{aligned}\right\} \qquad (5.17)$$

then

$$\left\{\begin{aligned} L^3\Delta\theta_{zz} &= q^2[1 - \varphi(\psi - t)\Delta] \\ L^3\Delta\theta_{zt} &= q^2[K - \varphi\Delta] \\ L^3\Delta\theta_{tt} &= K^2 - \varphi^3(\psi - t)\Delta \end{aligned}\right\} \qquad (5.18)$$

The expressions (4.21) for $A_i$,

$$\left.\begin{aligned} qA_1 &= \operatorname{cosec}^2\alpha \ \tan\alpha\ \lambda_q \\ qA_2 &= \operatorname{cosec}^2\alpha \ \tan\alpha\ \lambda_t \\ qA_3 &= \tan\alpha\ \operatorname{cosec}\alpha(\varphi_\lambda + \varphi\cot\alpha)\lambda_t \\ qA_4 &= \varphi\tan\alpha\ \operatorname{cosec}\alpha \end{aligned}\right\} \tag{5.19}$$

become, with the aid of (5.17), and (5.15),

$$\left.\begin{aligned} L^2\Delta\varphi(\psi - t)A_1 &= q^2 \\ L^2\Delta\varphi(\psi - t)A_2 &= qK \\ L^2\Delta\varphi(\psi - t)A_3 &= K^2 \\ (\psi - t)A_4 &= 1 \end{aligned}\right\} \tag{5.20}$$

therefore
$$\frac{A_1}{q^2} = \frac{A_2}{qK} = \frac{A_3}{K^2} = \frac{A_4}{L^2\Delta\varphi} = \frac{1}{L^2\Delta\varphi(\psi - t)} \tag{5.21}$$

It is clear from (5.19) and (5.21) that,

$$\frac{A_3}{A_1} = \frac{K^2}{q^2} = \frac{\lambda_t^2}{\lambda_q^2}$$

and that, from (5.15),

$$K\lambda_q - q\lambda_t = 0$$

therefore
$$A_3\lambda_q^2 - A_1\lambda_t^2 = 0 \tag{5.22}$$

Also;

$$\left.\begin{aligned} Lq\theta_q &= \varphi(\psi - t) \\ L\theta_t &= \varphi \\ \underline{q}_q^2 &= 1 + q^2\theta q^2 = \frac{q^2}{L^2} \end{aligned}\right\} \tag{5.23}$$

<u>Relations Between $A_i, B_i$.</u> If $z = \log q$ then

$$\left\{\begin{array}{ll} q^2 A_1 = 1 + \theta_z^2 + \dfrac{\theta_{zz}}{\theta_z} & qA_2 = \dfrac{\theta_t(1+\theta_z^2)+\theta_{zt}}{\theta_z} \\[2ex] A_3 = \theta_t^2 + \dfrac{\theta_{tt}}{\theta_z} & A_4 = \dfrac{\theta_t}{\theta_z} \end{array}\right\}$$

$$\left\{\begin{array}{ll} -qB_1 = 1 + \dfrac{\theta_{zz}}{\theta_z} & -B_2 = \dfrac{\theta_t + \theta_{zt}}{\theta_z} \\[2ex] -\dfrac{1}{q}B_3 = \dfrac{\theta_{tt}}{\theta_z} & -\dfrac{1}{q} B_4 = \dfrac{\theta_t}{\theta_z} \end{array}\right\}$$

Hence

$$\left\{\begin{array}{ll} B_1 = q(\theta_q^2 - A_1) & B_2 = q(\theta_q\theta_t - A_2) \\ B_3 = q(\theta_t^2 - A_3) & B_4 = -qA_4 \end{array}\right\} \tag{5.24}$$

<u>There Exists the Following Identities.</u> If we eliminate $(\theta_q, \theta_t)$ between equations (5.24), using

$$(q\theta_q^2)(q\theta_t^2) = (q\theta_q\theta_t)^2$$

then

$$(qA_1 + B_1)(qA_3 + B_3) = (qA_2 + B_2)^2$$

that is

$$(B_1B_3 - B_2^2) + q^2(A_1A_3 - A_2^2) + q(A_1B_3 + A_3B_1 - 2A_2B_2) = 0$$

Also, directly,

$$\left\{\begin{array}{l} \dfrac{1}{q}(A_1B_3+A_3B_1-2A_2B_2) = -2(A_1A_3-A_2^2)+(A_1\theta_t^2-2A_2\theta_q\theta_t+A_3\theta_q^2) \\[2ex] \dfrac{1}{q^2}(B_1B_3-B_2^2) = (A_1A_3 - A_2^2) - (A_1\theta_t^2-2A_2\theta_q\theta_t+A_3\theta_q^2) \end{array}\right\} \tag{5.25}$$

<u>The Linear Equation for $\Phi(q;t)$.</u> The two Monge-Ampère equations (4.12), and (4.13) are both equations for $\Phi(q;t)$ in which the coefficients $A_i$, $B_i$ are functions of $(q,t)$ and $\Theta(q;t)$, the solution of (5.1) as expressed by the system (5.10), and (5.11). The final expressions for the coefficients $A_i$ are given in equation (5.21) in terms of arbitrary functions $\varphi(\lambda)$, $\psi(\lambda)$ of the variable $\lambda(q;t)$ (given by (5.12)) and the functions $L(\lambda;t)$, $\Delta(\lambda;t)$, $K(\lambda;t)$ (given by (5.16)). The equation (4.12) is a general second degree Monge-Ampère equation, but (4.13) is of the first degree in $\Phi(q;t)$. It is preferable therefore to study, first, solutions of (4.13) and then return to (4.12) to obtain any possible solutions common to both these equations.

The equation (4.13) may be written

$$A_3\Phi_{qq}-2A_2\Phi_{qt}+A_1\Phi_{tt}+(A_3B_1+A_1B_3-2A_2B_2)\Phi_q=2qA_2+(\gamma-1)A_4q_q^2$$

or, in virtue of (5.25), since $A_1A_3 = A_2^2$,

$$A_3\Phi_{qq}-2A_2\Phi_{qt}+A_1\Phi_{tt}+q\Phi_q(A_3\Theta_q^2-2A_2\Theta_q\Theta_t+A_1\Theta_t^2)=2qA_2+(\gamma-1)A_4q_q^2 \qquad (5.26a)$$

and, using (5.21), and (5.23)

$$K^2\Phi_{qq}-2qK\Phi_{qt}+q^2\Phi_{tt}+q\Phi_q(K\Theta_q-q\Theta_t)^2 = q^2\left\{2K+(\gamma-1)\varphi\Lambda\right\}$$

But from (5.23)

$$K\Theta_q - q\Theta_t = \frac{\varphi}{qL}[K(\psi - t) - q^2] = -\frac{\varphi^2\psi_\lambda}{q}$$

after simplification with (5.16).

The final equation for $\Phi$ is therefore

$$K^2\Phi_{qq} - 2qK\Phi_{qt} + q^2\Phi_{tt} + \varphi^4\psi_\lambda^2 \frac{1}{q} \varphi_q = q^2 \left\{2K + (\gamma - 1)\varphi\Delta\right\} \quad (5.26b)$$

This equation is <u>parabolic</u>, the coincident characteristics being given by

$$q^2dq^2 + 2qKdqdt + K^2dt^2 = 0$$

that is

$$qdq + Kdt = 0 \quad (5.26c)$$

which in virtue of (5.15) may be written

$$\lambda_q dq + \lambda_t dt = 0$$

Therefore the characteristics coincide with the curves

$$\lambda(q;t) = \text{constant} \quad (6.27)$$

Form (5.12), these $\lambda$-curves constitute the one parameter family of hyperbolas

$$(\varphi^2 + \varphi_\lambda^2)(\psi - t)^2 + 2\varphi\varphi_\lambda\psi_\lambda(\psi - t) + \varphi^2\psi_\lambda^2 = q^2$$

in the (q;t) plane. The subsidiary system of the equation (5.26) is

$$K^2dPdt + q^2dQdq = [q^2(2K + \overline{\gamma - 1}\ \varphi\Delta) - \varphi^4\psi_\lambda^2Pq^{-1}]dqdt$$

where $P = \Phi_q$; $Q = \Phi_t$. With the use of (5.26c), this system reduces to

$$-\left(\frac{K}{q}\right) dP + dQ + \frac{\varphi^4\psi_\lambda^2}{q^3} Pdt = (2K + \overline{\gamma - 1}\varphi\Delta)dt \quad (5.28)$$

Since $\lambda(q;t)$ = const. are the characteristic curves, it is possible to integrate the system (5.28) regarding $\lambda$, $\varphi(\lambda)$, $\psi(\lambda)$ as constants.

For this purpose (5.16), and (5.12) give

$$-\frac{dK}{dt} = \varphi^2 + \varphi_\lambda^2$$

$$-\frac{dq}{dt} = \frac{K}{q}$$

Then
$$q^2 \frac{d}{dt}\left(\frac{K}{q}\right) = q\frac{dK}{dt} - K\frac{dq}{dt}$$

therefore
$$q^3 \frac{d}{dt}\left(\frac{K}{q}\right) = K^2 - q^2(\varphi^2 + \varphi_\lambda^2) = -\varphi^4\psi_\lambda^2$$

Insertion into (5.28) enables the integration to be performed giving
$$Q - \frac{KP}{q} = \int[2K(\lambda,t) + \overline{\gamma - 1}\,\varphi(\lambda)\Delta(\lambda,t)]dt + C$$

where the indefinite integration is to be performed with λ constant, and C is a constant of integration. The intermediate integral of (5.26) is obtained by making C an arbitrary function of λ, therefore
$$\Phi_t - \frac{K}{q}\Phi_q = C(\lambda) + F(\lambda,t) \qquad (5.29)$$

where

$$-F(\lambda,t) = \tfrac{1}{2}[(\gamma+1)\varphi^2 + 2\varphi_\lambda^2 + (\gamma-1)\varphi\varphi_{\lambda\lambda}](\psi - t)^2$$
$$+ \varphi(2\gamma\varphi_\lambda\psi_\lambda + \overline{\gamma - 1}\,\varphi\psi_{\lambda\lambda})(\psi - t) \qquad (5.30)$$

This intermediate integral is linear in Φ and has the subsidiary system
$$\frac{dt}{1} = \frac{q\,dq}{-K} = \frac{d\Phi}{C(\lambda) + F(\lambda;t)}$$

in which $\lambda(q;t)$ is still given by (5.12). The first of the equations implies
$$q\,dq + K\,dt = C$$

which (cf. equation (5.26c)) defines the original characteristics

already encountered, and may be taken as a first integral of (5.29)

$$\lambda(q;t) = \text{constant}$$

and therefore this condition may be employed in conjunction with

$$\frac{d\Phi}{dt} = C(\lambda) + F(\lambda;t)$$

to yield the second integral of (5.29) (by partial integration with respect to time with $\lambda$ constant) in terms of a constant D,

$$\Phi(q;t) = - C(\lambda)(\psi - t) + \int F(\lambda;t)dt + D$$

The general integral of (5.29), and therefore the complete integral of (5.26), may be obtained by making D an arbitrary function of $\lambda$, so that

$$\Phi(q;t) = D(\lambda) - C(\lambda)(\psi - t) + E(\lambda)(\psi - t)^2 + G(\lambda)(\psi - t)^3 \quad (5.31)$$

where $C(\lambda)$, $D(\lambda)$ are arbitrary, and

$$G(\lambda) = \frac{1}{6}\left[(\gamma + 1)\varphi^2 + 2\varphi_\lambda^2 + (\gamma - 1)\varphi\varphi_{\lambda\lambda}\right]$$

$$E(\lambda) = \frac{1}{2}\varphi\left[2\gamma\varphi_\lambda\psi_\lambda + (\gamma - 1)\varphi\psi_{\lambda\lambda}\right]$$

<u>The Monge-Ampère Equation for $\Phi(q;t)$</u>. The relations (5.10), (5.11), and (5.31) define $\Theta(q;t)$; $\Phi(q;t)$ in terms of the arbitrary functions $\varphi(\lambda)$, $\psi(\lambda)$, $C(\lambda)$, $D(\lambda)$ of the parameter $\lambda(q;t)$ given implicitly in terms of $\varphi(\lambda)$, $\psi(\lambda)$ by means of

$$q^2 = (\varphi^2 + \varphi_\lambda^2)(\psi - t)^2 + 2\varphi\varphi_\lambda\psi_\lambda(\psi - t) + \varphi^2\psi_\lambda^2 \quad (5.32)$$

It is necessary to find, therefore, if it is possible, by suitable choice of these four arbitrary functions, to satisfy the additional

relation (4.12) between $\Phi$, $\Theta$. This relation

$$(\Phi_{qq} + B_1\Phi_q)(\Phi_{tt} + B_3\Phi_q) - (\Phi_{qt} + B_2\Phi_q + 2q)(\Phi_{qt} + B_2\Phi_q) - q^2 + (\gamma - 1)\left\{\frac{c^2 - q^2}{2} - \Phi_t - B_4\Phi_q\right\}(1 + q^2\Theta_q^2) = 0$$

may be written

$$\Phi_{qq}\Phi_{tt} - \Phi_{qt}^2 - 2q\Phi_{qt} + \Phi_q(B_3\Phi_{qt} - 2B_2 \cdot \Phi_{qt} + B_1\Phi_{tt}) - 2qB_2\Phi_q + (B_1B_3 - B_2^2)\Phi_q^2 - q^2 + (\gamma - 1)\left\{\frac{c^2 - q^2}{2} - \Phi_t - B_4\Phi_q\right\}(1 + q^2\Theta_q^2) = 0$$

Like equation (5.1) this is a Monge-Ampère equation for $\Phi(q;t)$ in which the coefficients are functions of $q$ and $\Theta(q;t)$ as well as $(\Phi_q;\Phi_t)$, since the second order terms of the second degree occur only through the presence of the Jacobian $\partial(\Phi_q;\Phi_t)/\partial(q;t) = \Phi_{qq}\Phi_{tt} - \Phi_{qt}^2$. This Monge-Ampère equation may be reduced slightly by using the relation (5.1) ($A_1A_3 = A_2^2$) for $\Theta(q;t)$ and the relation (5.26) for $\Phi(q;t)$.

Substitution of the expressions (5.24) for $B_i$ enables the equation for $\Phi$ to be written

$$\Phi_{qq}\Phi_{tt} - \Phi_{qt}^2 - 2q\Phi_{qt} + q\Phi_q(\Theta_t^2\Phi_{qq} - 2\Theta_q\Theta_t\Phi_{qt} + \Theta_q^2\Phi_{tt}) - q\Phi_q(A_3\Phi_{qq} - 2A_2\Phi_{qt} + A_1\Phi_{tt}) - 2q^2(\Theta_q\Theta_t - A_2)\Phi_q - q^2\Phi_q^2(A_1\Theta_t^2 - 2A_2\Theta_q\Theta_t + A_3\Theta_q^2) - q^2 + (\gamma - 1)\left\{\frac{c^2 - q^2}{2} - \Phi_t + qA_4\Phi_q\right\}(1 + q^2\Theta_q^2) = 0$$

Now using directly the linear equation for $\Phi$ (equation (5.26a)) this may be reduced to

$$\Phi_{qq}\Phi_{tt} = \Phi_{qt}^2 - 2q\Phi_{qt} + q\Phi_q(\theta_t^2\Phi_{qq} - 2\theta_q\theta_t\Phi_{qt} + \theta_q^2\Phi_{tt})$$

$$- q^2(1 + 2\theta_q\theta_t\Phi_q) + (\gamma - 1)\left\{\frac{c^2 - q^2}{2} - \Phi_t\right\}(1 + q^2\theta_q^2) = 0 \tag{5.33}$$

that is

$$(\Phi_{qq} + q\theta_q^2\Phi_q)(\Phi_{tt} + q\theta_t^2\Phi_q) - (\Phi_{qt} + q\theta_q\theta_t\Phi_q + q)^2$$
$$+ (\gamma - 1)\left\{\frac{c^2 - q^2}{2} - \Phi_t\right\}(1 + q^2\theta_q^2) = 0 \tag{5.34}$$

which does not contain, explicitly, any of the functions $(A_i, B_i)$ nor the second derivatives $(\theta_{qq}, \theta_{qt}, \theta_{tt})$ but only $(\theta_q, \theta_t)$.

The equation (5.33) (or its equivalent (5.34)) is not parabolic. The characteristics will be obtained by first finding the roots of the equation in $\mu$,

$$\mu^2 - 2q(1 + \theta_q\theta_t\Phi_q)\mu + q^2(1 + \theta_q\theta_t\Phi_q)^2 = (\gamma-1)\left\{\frac{c^2-q^2}{2} - \Phi_t\right\}(1+q^2\theta^2)$$

(again cf. Forsyth, Differential Equations, p. 485) that is, by the roots $\mu$,

$$[\mu - q(1 + \theta_q\theta_t\Phi_q)]^2 = (\gamma - 1)\left\{\frac{c^2 - q^2}{2} - \Phi_t\right\}(1 + q^2\theta_q^2)$$

Now the equation (5.34) will be parabolic only if

$$\Phi_t = \frac{1}{2}(c^2 - q^2)$$

or

$$\Phi(q;t) = \frac{1}{2}(c^2 - q^2)t + F(q)$$

but resubstitution into (5.34) gives, since $\Phi_{tt} = \Phi_{qt} + q = 0$,

$$\Phi_{qq}\Phi_q\theta_t = 0$$

$\Theta_t = 0$ is a degenerate (steady) flow whilst $\Phi_q$, $\Phi_{qq}$ cannot vanish and be simultaneously compatible with $\Phi_t = 1/2(c^2 - q^2)$.

In the general case it would be necessary to insert in (5.34) the previous solutions for $\Theta(q;t)$, $\Phi(q,t)$ in terms of the functions $\varphi(\lambda)$, $\psi(\lambda)$, $C(\lambda)$ and $D(\lambda)$ which must therefore be chosen to satisfy together the resulting equation. This procedure would be very complicated, however, owing to the expressions for $\Theta_q$, $\Theta_t$ and the derivatives of $\lambda(q;t)$ invoked during the substitution. Certain important consequences may be obtained from a direct consideration of the general solutions for $\Theta$ and $\Phi$ without any further regard to the conditions to be satisfied by the four functions $\varphi$, $\psi$, C, D. Therefore we shall not carry the general theory any further but merely state that any solution of a generalized simple wave must necessarily be of the form predicted by equations (5.10), and (5.31).

Concurrent Parameter Lines. If the lines $\Phi_q + \underline{r} \cdot \underline{q}_q = 0$ are concurrent through a fixed point, which without any restriction may be taken to coincide with the origin $\underline{r} = 0$, then $\Phi$ reduces to a function of time only and equations (5.26b) and (5.34) reduce to

$$\Phi_{tt} = 2K + (\gamma - 1)\varphi\Delta$$

$$q^2 = (\gamma - 1)\left\{\frac{c^2 - q^2}{2} - \Phi_t\right\}\underline{q}_q^2$$

where, from (5.23), $\underline{q}_q^2 = q^2/L^2$, therefore

$$\Phi_t = \frac{c^2 - q^2}{2} - \frac{L^2}{\gamma - 1}$$

Inserting in these expressions for $\Phi_t$; $\Phi_{tt}$ the values of L, K, $\Delta$ in terms of $\varphi(\lambda)$, $\psi(\lambda)$ as given by (5.16) gives

$$\Phi_{tt} = [(\gamma + 1)\varphi^2 + 2\varphi_\lambda^2 + (\gamma - 1)\varphi\varphi_{\lambda\lambda}](\psi - t) + 2\gamma\varphi\varphi_\lambda\psi_\lambda + (\gamma-1)\varphi^2\psi_{\lambda\lambda}$$

$$\Phi_{tt} = \frac{2\varphi_\lambda}{\gamma - 1}\left\{\varphi_\lambda(\psi - t) + \varphi\psi_\lambda\right\}$$

Now assuming that $\lambda$ does not reduce to a function of t only (otherwise (5.12) will give the contradiction that q is a function of t) the above simultaneous expressions for $\Phi_{tt}$, which is a function of t only, must simply be of the form (a + bt), a,b being constants independent of $\lambda$ or t.

These give the four conditions, in terms of constants $(\alpha,\beta)$,

$$(\gamma + 1)\varphi^2 + 2\varphi_\lambda^2 + (\gamma - 1)\varphi\varphi_{\lambda\lambda} = \alpha$$

$$2\gamma\varphi_\lambda\psi_\lambda + (\gamma - 1)\varphi\psi_{\lambda\lambda} = \beta$$

$$2\varphi_\lambda^2 = (\gamma - 1)\alpha$$

$$2\varphi\varphi_\lambda\psi_\lambda = (\gamma - 1)\beta$$

These equations are clearly incompatible. The third gives at once $\varphi_{\lambda\lambda} = 0$ and

$$\varphi(\lambda) = \pm\lambda\sqrt{\frac{\alpha(\gamma - 1)}{2}} + \varphi_o$$

which enables the first to be reduced to

$$(\gamma + 1)\left\{\varphi_o \pm \lambda\sqrt{\frac{\alpha(\gamma - 1)}{2}}\right\}^2 + (\gamma - 1)\alpha = \alpha$$

which is an impossible relation, even if $\alpha = 0$ then $\varphi_o = \varphi = 0$. It is concluded that there is no unsteady generalization of the simple wave for which conditions are instantaneously constant

along a series of straight lines radiating from a fixed point. However, this result says nothing about the possibility of lines concurrent through a moving point.

Oscillating Flows. The principal possible application of the preceding solution would be found if it were possible to choose the four functions $\varphi$, $\psi$, C, D in such a way that the time variation of the deflection function $\Theta(q;t)$ was harmonic. Thus consider a flat plate whose equation is written in terms of the parameters a(t), $\hat{\underline{n}}(t)$ (the unit normal vector) in the form

$$a + \underline{r} \cdot \underline{n} = 0 \tag{5.35}$$

then the boundary condition to be satisfied on this plate is

$$\left(\frac{\partial}{\partial t} + \underline{q} \cdot \nabla\right)(a + \underline{r} \cdot \underline{n}) = 0$$

giving

$$\dot{a} + \underline{q} \cdot \underline{n} + \underline{r} \cdot \dot{\underline{n}} = 0 \tag{5.36}$$

In addition there is the field equation, valid everywhere,

$$\Phi_q + \underline{r} \cdot \underline{q}_q = 0 \tag{5.37}$$

In the first place it is noted that the line (5.35) cannot instantaneously coincide with a parameter line of the family (5.37), except for the case when (5.35) simply represents a translational oscillation of the line with no pitching (i.e. $\underline{n}$ = a constant vector). For otherwise $(q,\Theta)$ would not be constant along such a line, as required by the boundary condition (5.36) when $\dot{\underline{n}} = 0$. Thus in the general case it is possible to eliminate $\underline{r}$ from the equations (5.35), (5.36), and (5.37) to give

$$\Phi_q(\underline{n}_\Lambda \dot{\underline{n}}) = - a(\dot{\underline{n}}_\Lambda \underline{q}_q) + (\dot{a} + \underline{n} \cdot \underline{q})(\underline{n}_\Lambda \underline{q}_q) \qquad (5.40)$$

If $n = (\cos\alpha, \sin\alpha)$; $\alpha = \alpha(t) = \omega t$ and

$$\underline{q} = (q\cos\Theta,\ q\sin\Theta)$$

then

$$\underline{q}_q = (\cos\Theta - q\Theta_q\sin\Theta,\quad \sin\Theta + q\Theta_q\cos\Theta)$$

Therefore

$$|\underline{n}_\Lambda \underline{q}_q| = \sin(\Theta - \omega t) + q\Theta_q\cos(\Theta - \omega t)$$

$$\underline{n} \cdot \underline{q} = q\cos(\Theta - \omega t)$$

$$\dot{\underline{n}}_\Lambda \underline{q}_q = - \omega[\cos(\Theta - \omega t) - q\Theta_q\sin(\Theta - \omega t)]$$

the required condition (5.40) becomes

$$\omega\Phi_q = \omega a[\cos(\Theta - \omega t) - q\Theta_q\sin(\Theta - \omega t)]$$

$$- [\dot{a} + q\cos(\Theta - \omega t)]-[\sin(\Theta - \omega t) + q\Theta_q\cos(\Theta - \omega t)]$$

Now

$$q\Theta_q = \frac{\varphi(\psi - t)}{L} = \cot(\Theta - \lambda)$$

$$q\cos(\Theta - \lambda) = \varphi(\psi - t)$$

$$q\sin(\Theta - \lambda) = L$$

and if $a = a_o\sin\omega t$ then

$$\omega\Phi_q = - \omega a_o[\sin(\Theta - 2\omega t) + q\Theta_q\cos(\Theta - 2\omega t)]$$

$$- q\cos(\Theta - \omega t)[\sin(\Theta - \omega t) + q\Theta_q\cos(\Theta - \omega t)]$$

finally, on inserting $q\Theta_q = \cot(\Theta - \lambda)$,

$$\sin(\theta - 2\omega t) + q\theta_q\cos(\theta\ 2\omega t) = \frac{\cos(\lambda - 2\omega t)}{\sin(\theta-\lambda)} = \frac{q}{L}\cos(\lambda - 2\omega t)$$

$$\sin(\theta - \omega t) + q\theta_q\cos(\theta - \omega t) = \frac{q}{L}\cos(\lambda - \omega t)$$

$$q\cos(\theta - \omega t) = \varphi(\psi - t)\cos(\lambda - \omega t) - L\sin(\lambda - \omega t)$$

therefore since $\Phi_q = \Phi_\lambda\lambda_q = \Phi_\lambda(\frac{q}{L\Delta})$

$$-\omega\Phi_\lambda\Delta^{-1} = \omega a_o\cos(\lambda - 2\omega t) + \cos(\lambda - \omega t)\left\{\varphi(\psi - t)\cos(\lambda - \omega t) - L\sin(\lambda - \omega t)\right\}$$

giving

$$-\omega\Phi_\lambda\Delta^{-1} = [\omega a_o\cos\lambda - \tfrac{1}{2}L\sin 2\lambda + \tfrac{1}{2}\varphi(\psi - t)\cos 2\lambda]\cos 2\omega t + [\omega a_o\sin\lambda + \tfrac{1}{2}L\cos 2\lambda + \tfrac{1}{2}\varphi(\psi - t)\sin 2\lambda]\sin 2\omega t + \tfrac{1}{2}\varphi(\psi - t) \quad (5.41)$$

Now having regard that the expressions (5.31) for $\Phi$ and (5.16) for $\Delta$ are simply (finite) polynomials in the time with coefficients functions of $\lambda$ , a representation of the type (5.41) for $\Phi$ can only be possible if both the factors of the sin $2\omega t$, cos $2\omega t$ terms vanish identically in $\lambda$. It is clear that this is never possible except in the trivial case $a_o = \varphi = 0$, giving also $q = L = \Delta = 0$.

In the exceptional case when $\dot{\underline{n}} = 0$ then (5.35), (5.37) give, since they must represent instantaneously coincident straight lines,

$$a\underline{q}_q = \Phi_q\underline{n}$$

where $\underline{n}$ is a fixed vector. Therefore $\underline{q}_q$ is parallel to a first vector and the system of lines of constant velocity, lines (5.37),

constitute a system of lines perpendicular to the fixed direction of $\underline{n}$. The required expression for $\Phi(q;t)$ then reduces to, if f(t) is an arbitrary function,

$$\Phi(q;t) = - \frac{q^2}{2} \cdot \frac{a(t)}{a'(t)} + f(t)$$

Possilbe flows falling into this pattern are dealt with in the next section.

6. Parallel Parameter Lines. This section considers the class of flows which fall into the analytic form $\underline{q}(\lambda;t)$ but which were automatically excluded in the analysis following equation (5.10). Such flows will be shown to be characterized by the velocity being constant along a series of parallel straight lines parallel to a fixed direction, and the component of velocity in this direction is instantaneously constant throughout the flow field and is a function of time only. These represent a very restricted class of flows and it is concluded that the previous analysis is always applicable, except in the present cases for which the parameter lines degenerate into a system of parallel lines. For completeness the present analysis shows the procedure to be adopted in this special case,but details of the flows are not discussed.

The analysis following (5.10) was performed on the assumption that the arbitrary function $\varphi(\lambda)$ occurring in (5.7) did not reduce to a constant. Otherwise it is not possible to take $\varphi(\lambda)$ as the current parameter to replace $\lambda$ and rewrite (5.7) in the form of (5.10). If $\varphi(\lambda)$ is a constant, which may

clearly be taken as zero, then (5.7) becomes

$$q \cos \theta = (\lambda - t)\psi(\lambda) \tag{6.1}$$

from which a more general solution for $\theta(q;t)$ is obtained by partial differiatiation with respect to $\lambda$,

$$0 = \psi(\lambda) + (\lambda - t)\psi'(\lambda) \tag{6.2}$$

and eliminating $\lambda$ between (6.1) and (6.2). Equation (6.2) implies simply that, since $\psi(\lambda)$ is arbitrary, $\lambda$ is an arbitrary function of time. Therefore (6.1) gives

$$q \cos \theta = q_o(t) \tag{6.3}$$

$q_o(t)$ being an arbitrary function of t. It follows by direct differentiation that,

$$\theta_z = q\theta_q = \cot \theta, \qquad q\theta_t = - q_o'(t)\operatorname{cosec} \theta \tag{6.4}$$

$$\theta_{zz} = - \operatorname{cosec}^2\theta \cot \theta$$

$$q\theta_{zt} = q_o'(t)\operatorname{cosec}^3\theta$$

$$q^2\theta_{tt} = - qq_o''(t)\operatorname{cosec} \theta - q_o'(t)^2\operatorname{cosec}^2\theta \cot \theta$$

The expressions for $A_i$, $B_i$ become

$$A_1 = A_2 = 0, \qquad A_3 = \frac{- q_o''(t)}{q_o(t)} ; \qquad A_4 = \frac{- q_o'(t)}{q_o(t)}$$

Since $A_1 = A_2 = 0$ it follows from (4.8) that

$$\theta + \tan^{-1} q\theta_q = \text{constant}$$

and the constant is not even a function of the time. Therefore the isovels constitute, at all times, a system of parallel lines parallel to a fixed direction. These flows could then clearly be obtained by a more direct method in which the velocity (but not the potential or the acoustic speed) is independent of one coordinate x(or y).

When the above values for $A_i$ are inserted in the linear equation for $\Phi$; equation (5.26a) it reduces to

$$\Phi_{qq} + \frac{q_o^2}{q(q^2 - q_o^2)} \Phi_q = \frac{(\gamma - 1)q^2}{q^2 - q_o^2} \cdot \frac{q_o'(t)}{q_o''(t)}$$

that is

$$\frac{\partial}{\partial q} \left[\frac{1}{q} \sqrt{q^2 - q_o^2} \cdot \Phi_q\right] = \frac{(\gamma - 1)q}{\sqrt{q^2 - q_o^2}} \cdot \frac{q_o'(t)}{q_o''(t)}$$

therefore

$$\Phi_q = (\gamma - 1) \frac{q q_o'(t)}{q_o''(t)} + \frac{qG(t)}{\sqrt{q^2 - q_o^2}}$$

finally

$$\Phi(q;t) = (\gamma - 1) \frac{q^2 q_o'(t)}{2q_o''(t)} + G(t) \sqrt{q^2 - q_o^2} + H(t) \qquad (6.5)$$

in terms of functions of integration G(t); H(t).

These functions G(t); H(t) are not arbitrary however, since (6.5) must satisfy, if possible, the non-linear equation (5.34). To express $\Phi(q;t)$ in terms of (y;t) resort must be made to the fundamental field equation, which relates the two systems,

$$\Phi_q + \underline{r} \cdot \underline{q}_q = 0$$

Since

$$\underline{q} = (q \cos \theta, q \sin \theta) = [q_o(t); q \sin \theta]$$

then

$$\underline{q}_q = [0 \; ; \; \sin\Theta + q\Theta_q \cos\Theta]$$

that is

$$\underline{q}_q = [0 \; ; \; \operatorname{cosec}\Theta]$$

in virtue of (6.4). Therefore the field equation reduces to, simply,

$$y = - \Phi_q \sin\Theta$$

or, substituting for $\Theta(q;t)$ from (6.3),

$$y = - \frac{1}{q} \sqrt{q^2 - q_o^2} \cdot \Phi_q \tag{6.6}$$

Equations (6.6), and (6.5) may be used to obtain the function $q(y;t)$ explicitly; since

$$y = - (\gamma - 1) \sqrt{q^2 - q_o^2} \cdot \frac{q_o'(t)}{q_o''(t)} - G(t)$$

therefore

$$q^2 = q_o^2 + \frac{q_o''(t)^2}{(\gamma - 1)^2 q_o'(t)^2} [y + G(t)]^2 \tag{6.7}$$

If this expression for $q = q(y;t)$ is inserted in (6.5) an expression of $\Phi$ as a function of $(y;t)$ will be obtained. This representation is of the form

$$\Phi(y;t) = - \frac{1}{2} y^2 A(t) + B(t) \tag{6.8}$$

where $A(t)$; $B(t)$ [being given in terms of $q_o(t)$; $G(t)$; $H(t)$] are not arbitrary but must satisfy the equivalent of equation (5.34), when the q-derivatives of $\Phi$ are transformed into ones with respect to y.

Returning to the original equation for $\underline{q}(y;t)$;

$$(a^2 - u^2)u_x - 2uvu_y + (a^2 - v^2)v_y = 2(u\dot{u} + v\dot{v}) + \ddot{\varphi}$$

this reduces to, since $u = u(t)$ $[= q_o(t)]$ and $v = v(y;t)$,

$$(a^2 - v^2)v_y = 2(u\dot{u} + v\dot{v}) + \ddot{\varphi} \qquad (6.9)$$

It should be recalled that the condition for irrotational flow $v_x = u_y$, is automatically satisfied by this system $[u(t), v(y;t)]$.

Also

$$\frac{a^2}{\gamma - 1} = \frac{1}{2}(c^2 - q^2) - \dot{\varphi}$$

and

$$\varphi = \Phi + xu(t) + yv(y;t)$$

therefore,

$$\Phi_x = 0; \quad \Phi_y + yv_y = 0$$

and

$$\dot{\varphi} = \Phi_t + xu_t + yv_t$$

$$\ddot{\varphi} = \Phi_{tt} + xu_{tt} + yv_{tt}$$

Substitution into (6.9) gives

$$(\gamma - 1)[\tfrac{1}{2}(c^2 - u^2 - v^2) - \Phi_t - xu_t - yv_t]v_y$$

$$- v^2 v_y = 2(uu_t + vv_t) + \Phi_{tt} + xu_{tt} + yv_{tt} \qquad (6.10)$$

u, v, $\Phi$ are independent of x, therefore

$$(\gamma - 1)u_t v_y + u_{tt} = 0$$

But u contains t only, therefore,

$$v_y = -\frac{u_{tt}}{(\gamma - 1)u_t} = A(t), \quad \text{say}$$

so that

$$v(y;t) = yA(t) + D(t) \tag{6.11}$$

$$u(t) = \int \exp\left[-(\gamma - 1)\int A(t)dt\right]dt \tag{6.12}$$

When (6.8), and (6.11) are inserted in (6.10), there results a quadratic expression in y and the coefficients, which are functions of t only, must be chosen so that the equation vanishes identically. This will yield three equations for A(t); B(t); D(t). [The functions A(t) occurring in (6.8), and (6.11) are identified in virtue of $\Phi_y + yv_y = 0$.] These equations are

$$A_{tt} + (\gamma + 3)AA_t + (\gamma + 1)A^3 = 0 \tag{6.13}$$

$$D_{tt} + (\gamma + 1)A(D_t + AD) + 2DA_t = 0 \tag{6.14}$$

$$(\gamma - 1)A[\tfrac{1}{2}(c^2 - u^2 - D^2) + B_t] = AD^2 + 2DD_t - B_{tt} + 2uu_t \tag{6.15}$$

The first equation (6.13) determines A(t) without reference to the remaining functions D(t), B(t). Let $\lambda = 2/(\gamma + 1)$ and rewrite (6.13),

$$\lambda A_{tt} + 2(\lambda + 1)AA_t + 2A^3 = 0 \tag{6.16}$$

which may be factorized into either of the forms

$$(2A + \lambda \frac{d}{dt})(A^2 + \frac{dA}{dt}) = 0 \tag{6.17}$$

$$(2A + \frac{d}{dt})(A^2 + \lambda \frac{dA}{dt}) = 0 \tag{6.18}$$

To solve write $p = dA/dt$; $p\, dp/dA = d^2A/dt^2$ and $\omega = A^2/p$ so that

$$2AdA = pd\omega + \omega dp$$

then (6.16) becomes

$$(\omega + 1)(\omega + \lambda)dp + (1 + \omega + \lambda)d\omega = 0$$

giving, on integration in terms of a constant $\mu$, and resubstituting $a^2 = p\omega$; $p = A_t$;

$$(A^2 + A_t)^\lambda = \mu(A^2 + \lambda A_t) \tag{6.19}$$

Since $\lambda = 5/6$ for $\gamma = 7/5$, attention would have to be confined to the special values of $\mu = 0, \infty$, since it is not clear how to integrate again the first order equation (6.19). However a parametric solution of (6.16), in which A and t are expressed in terms of a common parameter z, may be obtained which avoids resorting to (6.19).

Let $A = \psi_t/\psi$ then $A_t + A^2 = \psi_{tt}/\psi$ so that (6.17) yields

$$(2\psi_t + \lambda\psi \frac{d}{dt})(\frac{\psi_{tt}}{\psi}) = 0$$

that is

$$\lambda \frac{\psi_{ttt}}{\psi_{tt}} + (2 - \lambda) \frac{\psi_t}{\psi} = 0$$

therefore, if $\mu$, $\psi_o$ are constants,

$$\psi_t^2 = \mu^2 \left\{ \psi^{\frac{2(\lambda-1)}{\lambda}} - \psi_o^{\frac{2(\lambda-1)}{\lambda}} \right\}$$

When $\gamma = 7/5$; $\lambda = 5/6$ therefore

$$\psi_t^2 = \mu^2(\psi^{-2/5} - \psi_o^{-2/5})$$

If $\psi = \psi_o \sin^5\theta$ then

$$\mu t = 5\psi_o^{6/5} \int \sin^5\theta d\theta$$

giving

$$3\mu t = -\psi_o^{6/5}\cos\theta(3\cos^4\theta - 10\cos^2\theta + 15) \tag{6.20}$$

or

$$3\mu t = -\psi_o^{6/5} z(3z^4 - 10z^2 + 15) \tag{6.21}$$

where $z = \cos\theta$, $\psi = \psi_o(1 - z^2)^{5/2}$.

It remains to express A also in terms of $\theta$,

$$A = \frac{\psi_t}{\psi} = 5\cot\theta\theta_t$$

where, from (6.20),

$$5\psi_o^{6/5} \sin^5\theta\theta_t = \mu$$

so that

$$A(\theta) = \mu\psi_o^{-6/5}\cos\theta \operatorname{cosec}^6\theta$$

Therefore the solution of (6.16) may be arranged in the form

$$\left.\begin{aligned} A &= \alpha\cos\theta\operatorname{cosec}^6\theta \\ \alpha t &= -5\cos\theta(1 + \tfrac{1}{5}\cos^4\theta - \tfrac{2}{3}\cos^2\theta) \end{aligned}\right\} \tag{6.22}$$

where $\alpha(=\mu\psi_o^{-6/5})$ is an arbitrary constant.

When these results are inserted in (6.14) in which D is now regarded as a function of $\theta$ so that

$$5\sin^5\theta\, D_t = \alpha D_\theta$$

$$25\sin''\theta\, D_{tt} = \alpha^2(\sin\theta D_{\theta\theta} - 5\cos\theta\, D_\theta)$$

then

$$D_{\theta\theta} + 7\cot\theta D_\theta - 10 D = 0 \tag{6.23}$$

Let $z = \cos\theta$ then the equation for D becomes

$$(1 - z^2)D_{zz} - 8z\, D_z - 10 D = 0 \tag{6.24}$$

Finally (6.24) reduces to the hypergeometric equation by the substitution $x = z^2$,

$$x(1 - x)D_{xx} + \frac{1}{2}(1 - 9x)D_x - \frac{5}{2}D = 0 \qquad (6.25)$$

with general solution, with the usual notation for the hypergeometric functions,

$$D(x) = a_1 F(1, \tfrac{5}{2}, \tfrac{1}{2}, x) + a_2 x^{1/2} F(\tfrac{3}{2}, 3, \tfrac{3}{2}, x)$$

For the given values, the two independent solutions reduce to the finite forms

$$\left.\begin{aligned} F(\tfrac{3}{2}, 3, \tfrac{3}{2}, x) &= (1 - x)^{-3} \\ F(1, \tfrac{5}{2}, \tfrac{1}{2}, x) &= -\tfrac{1}{3}(x^2 - 6x - 3)(1 - x)^{-3} \end{aligned}\right\} \qquad (6.26)$$

Therefore, in terms of $\theta$, where $x = \cos^2\theta$,

$$D(\theta) = a_2 \cos\theta \operatorname{cosec}^6\theta + a_1(\cos^4\theta - 6\cos^2\theta - 3)\operatorname{cosec}^6\theta \qquad (6.27)$$

The first independent solution of (6.23) is, by comparison with (6.22), just a constant multiple of $A(\theta)$. This is otherwise obvious since if $D(t) = \text{constant} \times A(t)$ then equations (6.13), and (6.14) become identical and reduce to

$$D_{tt} + (\gamma + 3)DD_t + (\gamma + 1)D^3 = 0$$

Knowing that one solution of (6.24) is simply $z(1 - z^2)^{-3}$ it is possible to evade the general method (which leads to the hypergeometric equation (6.25) and the reduction then necessary to deduce the finite forms (6.26) from the original infinite

series) by seeking a solution

$$D(z) = \omega(z)z(1 - z^2)^{-3}$$

where $\omega(z)$ is to be determined by direct insertion into (6.24). This gives for $\omega(z)$

$$\frac{\omega_{zz}}{\omega_z} = -\frac{2(1 + z^2)}{z(1 - z^2)}$$

then

$$\omega(z) = \frac{K}{z}(z^4 - 6z^3 - 3) + K'$$

if K, K' are constants. Therefore for the second independent solution of (6.24) we can take

$$D(z) = (z^4 - 6z^3 - 3)(1 - z^2)^{-3}$$

which reproduces the second term of (6.27) with $z = \cos\theta = \sqrt{x}$. When also expressed in terms of z,

$$\left.\begin{aligned} A &= \alpha z(1 - z^2)^{-3} \\ \alpha t &= -\frac{1}{3} z(3z^4 - 10z^2 + 15) \\ \alpha\frac{dt}{dz} &= -5(1 - z^2)^2 \end{aligned}\right\} \qquad (6.29)$$

<u>The Flow in the Physical Plane</u>. Attention will be concentrated on the functions given by

$$A(t) = \alpha z(1 - z^2)^{-3}; \quad D(t) = \beta z(1 - z^2)^{-3}$$

where

$$\alpha t = - \frac{z}{3}(3z^4 - 10z^2 + 15)$$

then clearly for $v(y;t)$ in (6.11) we can take, with $\beta = -\alpha y_o$,

$$v = \alpha z(y - y_o)(1 - z^2)^{-3} \tag{6.30}$$

$u(t)$ is given by (6.12) rewritten, in virtue of (6.29),

$$\frac{d}{dz}(\log u_t) = - \frac{2A}{5}\frac{dt}{dz} = 2z(1 - z^2)^{-1}$$

therefore on integration in terms of parameters $(\mu, u_o)$,

$$u - u_o = \alpha\mu z(z^2 - 3) \tag{6.31}$$

The formulae (6.30), and (6.31) represent the complete parametric solution to the motion; z being given in terms of t by (6.29). To obtain the instantaneous streamlines (Fig. 3) defined by

$$\frac{dy}{dx} = \frac{v}{u}$$

use (6.30), and (6.31) then

$$\frac{dy}{dx} = (y - y_o)g(z)$$

(where $g(z) = \dfrac{-1}{\mu(3 - z^2)(1 - 2^2)^3}$ for $\mu_o = 0$). By integration the streamlines are obtained

$$y - y_o = e^{(x-x_o)g(z)} \tag{6.32}$$

since z is a function of time only.

To obtain the particle paths, the simultaneous equations

$$\frac{dx}{dt} = u(z); \qquad \frac{dy}{dt} = v(y;t)$$

in which z(t) is known, must be integrated.

Write

$$\frac{dy}{dz} = v(y;z)\frac{dt}{dz} = -5z(y - y_o)(1 - z^2)^{-1}$$

after using (6.30), and (6.29). On integration this yields

$$1 - z^2 = K(y - y_o)^{2/5}, \qquad (6.33)$$

K being a constant.

Also

$$\frac{dx}{dz} = u(z)\frac{dt}{dz} = -\frac{5}{\alpha}[u_o + \alpha\mu z(z^2 - 3)](1 - z^2)^2$$

so that x may be expressed as a polynomial in z, which, in conjunction with (6.33), will provide the particle paths in terms of z as current parameter.

If $u_o = 0$ then

$$\frac{dx}{dz} = -5\mu z(z^2 - 3)(1 - z^2)^2$$

and

$$\mu\frac{dy}{dx} = \frac{y - y_o}{(z^2 - 3)(1 - z^2)^3}$$

Substituting for $z^2$ from (6.33) gives

$$-\mu K^3\frac{dy}{dx} = (y - y_o)^{-1/5}\left\{2 + K(y - y_o)^{2/5}\right\}^{-1}$$

therefore

$$-\frac{x}{5\mu K^3} = \frac{1}{3}(y - y_o)^{6/5} + \frac{K}{8}(y - y_o)^{8/5} + K' \tag{6.34}$$

the constants of integration (K, K') being used to specify the $\infty^2$ particles, by, for example, their positions at some specified time. We will consider $\alpha < 0$, $\mu < 0$. On putting $\eta^5 = y - y_o$ the particle paths may be written parametrically in terms of $\eta$ to rationalize (6.34),

$$-\frac{24x}{5\mu K^3} = \eta^6(8 + 3K\eta^2) + K'$$

where $\mu$ is a constant for the flow as a whole but where K, K' are used to parameterize the $\infty^2$ particles. The fluid particle (K,K') is situated at $[x(\eta), y(\eta)]$ at the time $t(\eta)$ given by

$$3\alpha t = -\sqrt{1 - K\eta^2} \cdot (3K^2\eta^4 + 4K\eta^2 + 8)$$

(obtained by eliminating z between (6.29) and (6.33). In the present solution $|z| \leq 1$ therefore, since $1 - z^2 = K\eta^2$, the parameter K must be positive for all particles. It follows that the current parameter $\eta$ is confined to the range $(-K^{1/2}, +K^{1/2})$. By first drawing the separate curves of (x,y) against $\eta$ and afterwards combining them to obtain the particle paths given by the x against y curves. At zero time $\eta = K^{-1/2}$ and $z = 0$ therefore $u = v = 0$ everywhere. Thus the fluid is instantaneously at rest at zero time when the particles begin to advance towards the line $y = y_o$, arriving there at $t = -8/3\alpha$ ($> 0$) when $\eta = 0$. It is easily verified that

$$\frac{\partial u}{\partial t} = \frac{3\mu\alpha^2}{5K\eta^2}, \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} = -\frac{\alpha^2}{5K^6\eta^7}$$

therefore the acceleration becomes infinite when $\eta = 0$. The curves of x and y against $\eta$ and the final (x,y) curve for this solution are denoted by the full lines in Figures 12, 13 and 14.

The present analysis cannot yield a solution for $t > -8/3\alpha$ or for $y < y_o$.

In the equation preceding (6.20) it would be equally legitimate to use

$$\psi_t^2 = \mu^2(\psi^{-2/5} + \psi_o^{-2/5})$$

and the relevant equations now become, if $\alpha = \mu\psi_o^{-6/5}$

$$\psi = \psi_o(z^2 - 1)^{5/2}$$

$$A = \alpha z(z^2 - 1)^{-3}$$

$$3\alpha t = z(3z^4 - 10z^2 + 15)$$

$$\alpha\frac{dt}{dz} = 5(z^2 - 1)^2$$

$$v(y;t) = (y - y_o)A(t) = \alpha z(y - y_o)(z^2 - 1)^{-3}$$

$$u(t) = u_o + \alpha\mu\, z(z^2 - 3)$$

The instantaneous streamlines are now given by, when $u_o = 0$,

$$y - y_o = e^{+(x-x_o)g(z)}$$

$$\mu g(z) = (z^2 - 3)^{-1}(z^2 - 1)^{-3}$$

and the particle paths by

$$y - y_o = \eta^5, \quad z^2 - 1 = K\eta^2$$

$$\frac{24x}{5\mu K^3} = \eta^6(3K\eta^2 - 8) + K'$$

$$3\alpha t = -\sqrt{1 + K\eta^2} \cdot [3K^2\eta^4 - 4K\eta^2 + 8]$$

where [ $x(\eta)$, $y(\eta)$], $z(\eta)$ give the position and speed as above at time $t(\eta)$ of the particle $(K, K')$. We may choose $y_o = 0$ without loss of generality.

This second solution applies when $|z| \geq 1$ and therefore the parameter K must be positive for all particles since $z^2 - 1 = K\eta^2$. Unlike the first solution, however, the parameter may take on all real values. In the diagrams (Figs. 12, 13, and 14) the first solution is shown by the full lines and the second solution by the broken lines. The first solution ceases to apply when the time $t = |\frac{8}{3\alpha}|$ is reached, and the second solution only applies for $t > |\frac{8}{3\alpha}|$. Therefore a solution valid for $0 \leq t < \infty$ may be obtained by concentrating attention on the section OA( $\eta \geq 0$) of the first solution in conjunction with section OB ( $\eta \leq 0$) of the second. Initially only the upper half plane is occupied by fluid instantaneously at rest, corresponding to point A ($z = t = 0$, $\eta = K^{-1/2}$). When point O ($\eta = 0$) is reached the fluid is moving with velocity approaching $\infty$ and the acceleration is infinite at time $|\frac{8}{3\alpha}|$. If the second solution OB is then considered, the fluid concentrated along the x-axis expands into the lower half plane and eventually recedes to infinity parallel to this axis. The diagrams show only the motion of a typical value of K and when $K' = 0$. The constant K' merely serves to provide the x-wise position of the fluid elements at,

say, $t = 0$.

The phenomenon which takes place on the axis is not a simple shock. In the initial motion the advancing layers of fluid overtake themselves and cause a concentration of fluid moving at high speed across the axis, thereafter reducing speed during the subsequent motion in the lower half plane, but accelerating, again, to the right as $t \to \infty$.

<u>Conclusion</u>. The more important results of this work may be summarized as follows.

1. In unsteady potential flow of the simple wave pattern, the lines of constant $(q,\theta)$ are straight. However, in the general flow $\underline{q}(\lambda;t)$, the pressure and density and acoustic speed are not constant along such lines, except in the simplified case $\underline{q}(\lambda)$. In all cases these lines may constitute a system of parallel lines parallel to a fixed direction but may never be concurrent. In general the $\lambda$-lines envelope some curve, which grows linearly with time in the simplified wave.
2. In the general wave, the solution for $\Theta$ and $\Phi$ may be expressed in terms of four functions of the parameter $\lambda(q;t)$; these functions being chosen so that together they satisfy a Monge-Ampere equation for $\lambda(q;t)$. This representation is sufficient to show that the $\lambda$-lines can never be concurrent in the physical plane and that it is not possible to obtain any flows of an oscillatory nature.

These deductions are in contrast to the steady flow phenomena. For steady potential flow the $\lambda$-lines may be parallel, concurrent or possess a non-degenerate envelope. If the steady iso-energetic flows become rotational then the lines never possess a non-degenerate envelope and must either be parallel or concurrent. In contrast the latter alternative is the very case which is excluded when the simple wave becomes unsteady. It is useless therefore to attempt to obtain an unsteady simple centred wave, just as it is impossible to generalize the steady rotational simple centred wave.

## Bibliography


1. Prim, R. C. "On a Family of Rotational Gas Flows", Quart. Appl. Math., 1948.

2. Martin, M. H. "Steady Rotational Plane Flow of a Gas", Am. J. of Math. vol. 72 (1950). N.O.L.M. 10274.

3. Martin, M. H. "Steady Rotational Plane Prandtl-Meyer Flows", J. of Math. & Phys. vol. 29 (1950). N.O.L.M. 10346.

4. Martin, M. H. "Steady Plane Rotational Prandtl-Meyer Flows of a Polytropic Gas", J. Math. & Phys. vol. 29 (1951)

5. Ghaffari, A. "A Simplified Theory of Simple Waves", Aeronautical Quarterly, vol. 1 (1949).

6. Giese, J. "Compressible Flows with Degenerate Hodographs", Quart. Appl. Math. 1950.

7. Naylor, D. "The Simple Wave in Rotational Gas Flow", J. Rational Mech. & Analysis, vol. 3 (1954).

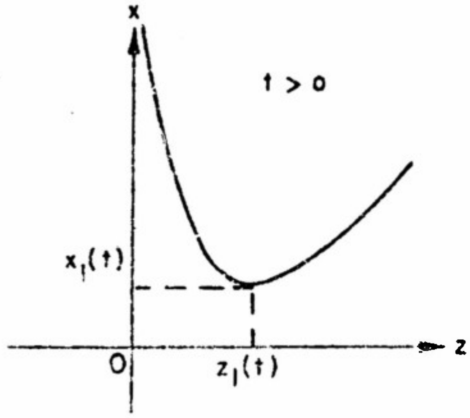


FIG. 1

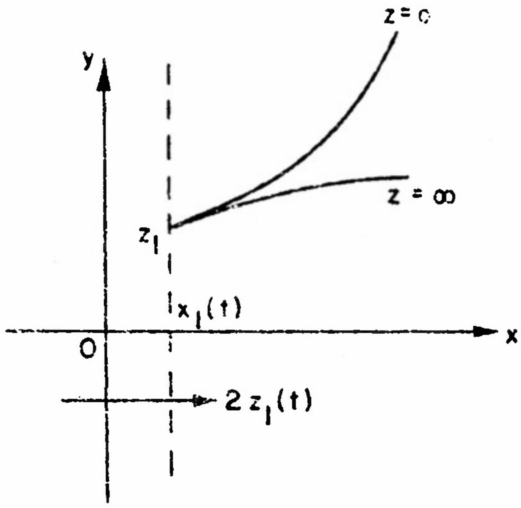


FIG. 2

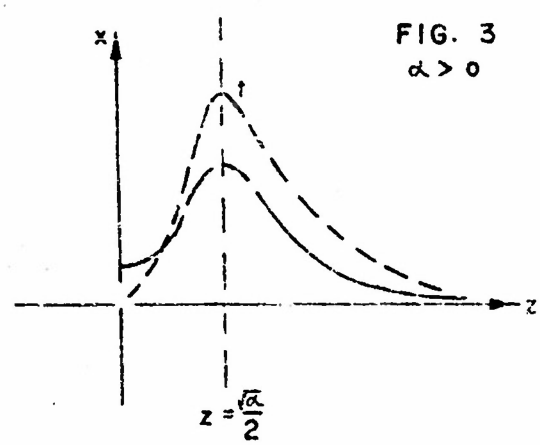


FIG. 3
$\alpha > 0$

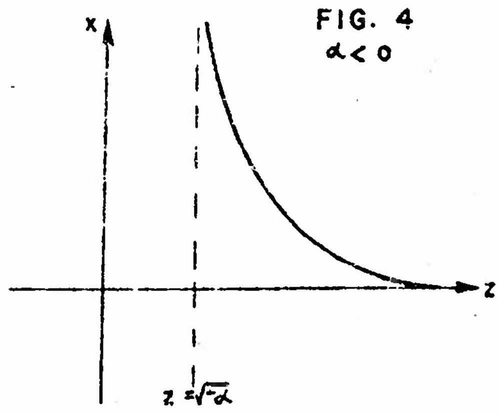


FIG. 4
$\alpha < 0$

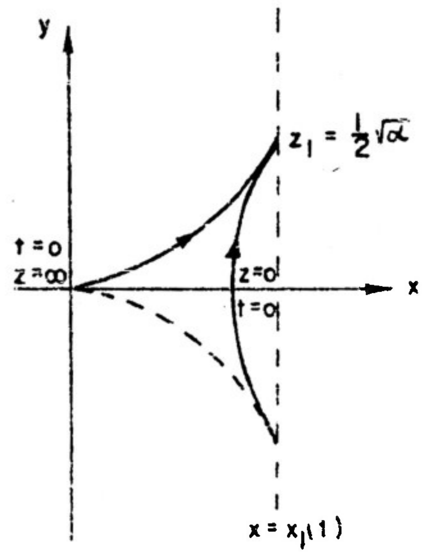


FIG. 5

$\alpha > 0$

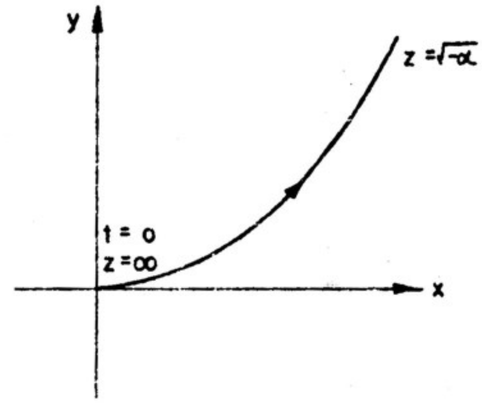


FIG. 6

$\alpha < 0$

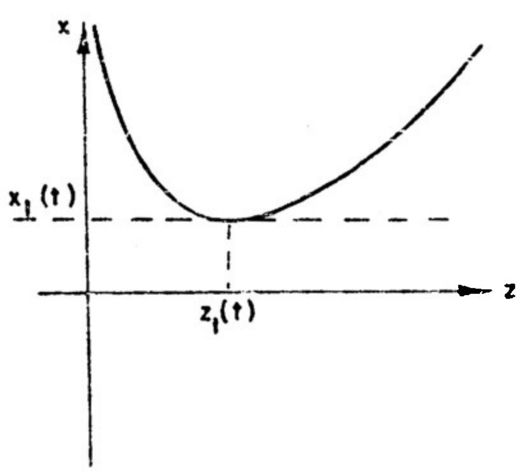


FIG. 7

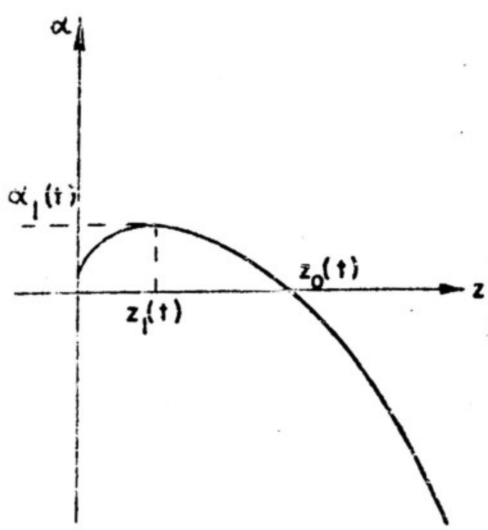


FIG. 8

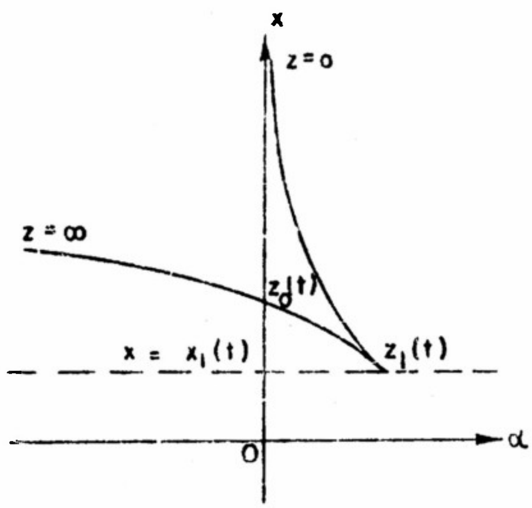


FIG. 9

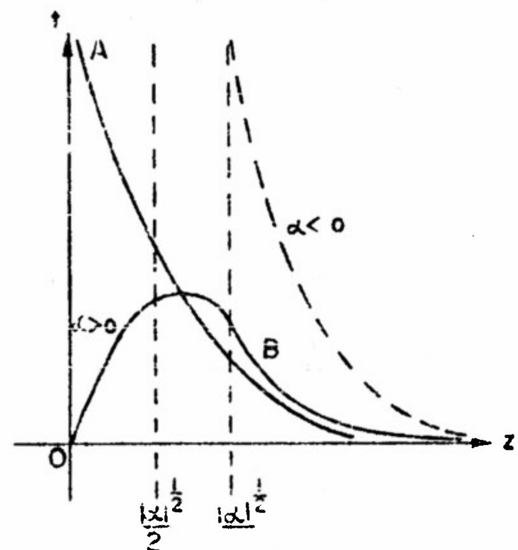


FIG. 10

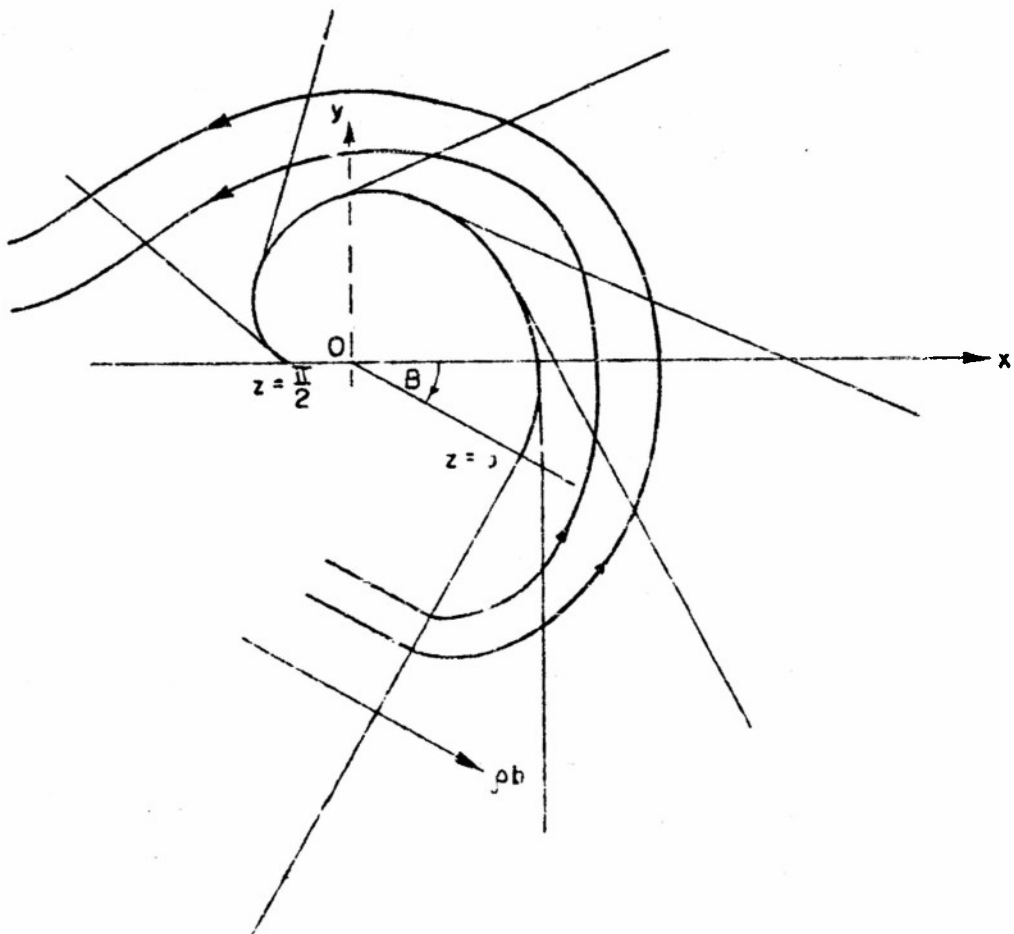


FIG. 11

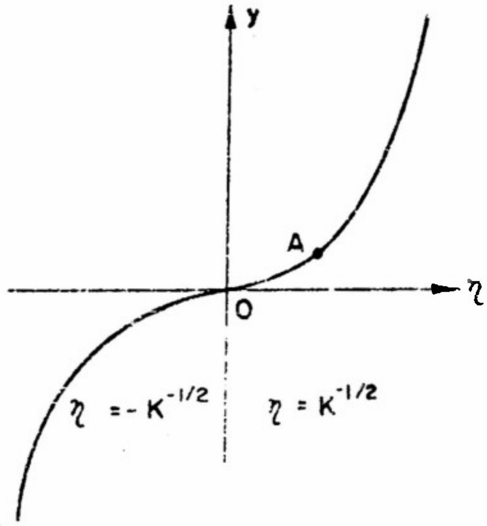


FIG. 12

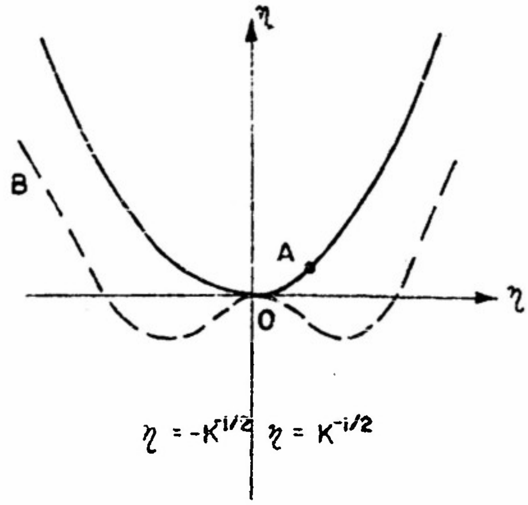


FIG. 13

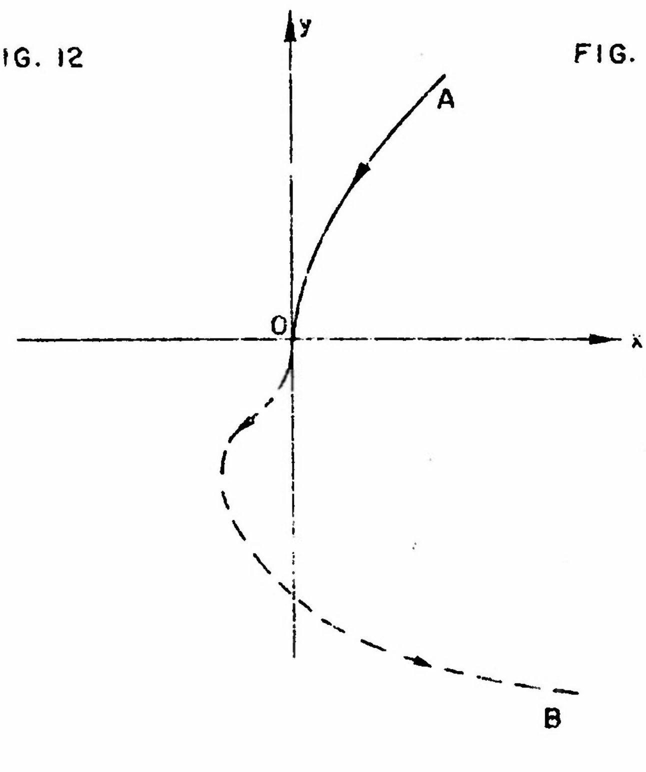


FIG. 14